

# DEFORMATIONS OF HYPERBOLIC 3-CONE-MANIFOLDS

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**ABSTRACT.** We show that any compact orientable hyperbolic 3-cone-manifold with cone angles at most  $\pi$  can be continuously deformed to a complete hyperbolic manifold homeomorphic to the complement of the singularity. This together with the local rigidity by Hodgson and Kerckhoff implies the global rigidity for compact orientable hyperbolic 3-cone-manifolds under the same angle assumption.

## 0. INTRODUCTION

A hyperbolic 3-cone-manifold is a riemannian 3-manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics (see [7, 8]). To each component of the singularity, associated is a cone angle. The cone angle is a positive real number. It possibly attains  $2\pi$ . In this particular case, the singular set is not singular and simply a finite union of disjoint simple closed geodesics. The hyperbolic 3-cone-manifold is a generalization of the hyperbolic 3-orbifold with vertexless singularity.

We are concerned with the deformations of a hyperbolic 3-cone-manifold with constant topological type. The hyperbolic Dehn filling theory by Thurston in [14], which describes deformations of a complete hyperbolic manifold in more wild setting, is a pioneering work of this subject. One important progress from a rather classical viewpoint was made recently by Hodgson and Kerckhoff [7]. They proved the local rigidity when cone angles are  $\leq 2\pi$ . It corresponds to Weil [18] and Garland [4] rigidity for hyperbolic manifolds of finite volume.

In [9], we showed as one of applications of the local rigidity that the underlying space of a nonsingular part of a compact hyperbolic 3-cone-manifold admits a complete hyperbolic structure of finite volume. We come back to this result later. The purpose of the present paper is to connect that complete structure with the original singular structure by a continuous family of cone-manifolds with constant topological type under the assumption that initial cone angles are  $\leq \pi$ . The main theorem is

**Theorem .** *Let  $C$  be a compact orientable hyperbolic 3-cone-manifold and  $\Sigma$  a singular set which forms a link in  $C$ . If the cone angles assigned to the components of  $\Sigma$  all are at most  $\pi$ , then  $C$  admits an angle decreasing continuous family of deformations to the complete hyperbolic manifold homeomorphic to the nonsingular part  $C - \Sigma$ .*

There are two immediate corollaries related to the representations in the group of orientation preserving isometries of the hyperbolic 3-space  $\mathbf{H}^3$ , isomorphic to

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$\mathrm{PSL}_2(\mathbf{C})$ . For cone-manifolds, we have a holonomy representation of a nonsingular part, so that meridional loops of the singularity are mapped to elliptic elements. The representation so obtained could be neither faithful nor discrete. Nevertheless, the local rigidity in [7] asserts that a neighborhood of this wild representation is parameterized up to conjugacy by Dehn filling coefficients [14] which are geometrically well understood.

One corollary is about the global rigidity. Weil and Garland rigidity states that any nearby discrete faithful representations of a group in  $\mathrm{PSL}_2(\mathbf{C})$  with finite volume quotients are conjugate. In the case of cone-manifolds, the local rigidity says in particular that nearby representations of a holonomy representation of a nonsingular part with constant rotation angles for meridians are conjugate. Mostow [11] and Prasad [12] rigidity for a hyperbolic manifold of finite volume then asserts that any discrete faithful representations in  $\mathrm{PSL}_2(\mathbf{C})$  are not only locally but globally conjugate each other. This global rigidity implies a geometric consequence that homeomorphic hyperbolic manifolds of finite volume are isometric. We state the global rigidity for hyperbolic 3-cone-manifolds rather in terms of this geometric terminologies.

**Corollary 1.** *Let  $C$  be a compact orientable hyperbolic 3-cone-manifold with singularity  $\Sigma$  where cone angles assigned to the components of  $\Sigma$  all are  $\leq \pi$ . If  $(C', \Sigma')$  is homeomorphic to  $(C, \Sigma)$  so that the corresponding cone angles all are the same, then  $C'$  and  $C$  are isometric.*

The other corollary is about liftability of  $\mathrm{PSL}_2(\mathbf{C})$ -representations into  $\mathrm{SL}_2(\mathbf{C})$ . The liftability has been discussed particularly for discrete subgroups in  $\mathrm{PSL}_2(\mathbf{C})$ . As was pointed out in [2], the liftability depends only on the component of the space of representations. The main theorem will be proved by connecting the holonomy representation of a cone-manifold with that of a complete structure by a particular path in the space of representations. Since the holonomy representation of a complete hyperbolic manifold is known to lift in [14], we have

**Corollary 2.** *The holonomy representation of a compact orientable hyperbolic 3-cone-manifold can be lifted to a  $\mathrm{SL}_2(\mathbf{C})$ -representation if the cone angles assigned all are at most  $\pi$ .*

It is quite unlikely that the angle assumptions in Theorem and Corollaries are necessary, though the argument we develop here uses its advantage. More progress should be expected.

A fairly large part of the proof of the main theorem is due to Thurston's strategy for the geometrization of orbifolds [15, 16] together with the local rigidity by Hodgson and Kerckhoff. The over all logic of Thurston's argument and some of its details can be found in [13]. We convey its minimal essentials for our purpose here, and hence the exposition will be reasonably self-contained.

This paper is organized as follows. In the first section, we will review some basic facts about hyperbolic 3-cone-manifolds. Also we improve the results in [9] from more Riemannian geometric viewpoints due to Steve Kerckhoff. The second section is to introduce two main machineries, the local rigidity and the pointed Hausdorff-Gromov topology. They are fundamental when cone angles are  $\leq 2\pi$ . The third section is to establish a few tools to control the local geometry of cone-manifolds away from the singularity when cone angles are  $\leq \pi$ . This section contains a technical but the most

crucial part of the analysis. In section 4, we establish a discrete setting of the problem. Then we study what happens when tubular neighborhoods of the singularity in the deformations are uniformly thick in section 5, and when otherwise in section 6. In the final section, we study what happens to continuous families and prove Theorem and Corollaries.

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In addition, the author would like to thank the referee for pointing out an earlier work by Qing Zhou [19] which discusses similar deformations. Zhou's argument together with the local rigidity now established by Hodgson and Kerckhoff [7] lead to another proof of the existence of an angle decreasing family in the main theorem provided that the initial cone angles all are at most  $2\pi/3$ . Also, Zhou proved Theorem 1.2.1 in slightly different manner.

## 1. HYPERBOLIC 3-CONE-MANIFOLDS

In this section, we assemble a few standard notions and notations which we use throughout this paper, improve the results in [9] and discuss an upper bound of the volume of hyperbolic 3-cone-manifolds with the same topological type.

**1.1. Preliminaries.** Let  $X$  be a metric space with a metric  $d$ . An  $R$ -neighborhood of  $x \in X$  for  $R > 0$  is the set of points in  $X$  from which the distance to  $x$  is  $< R$ , and denoted by

$$B_R(X, x) = \{y \in X \mid d(y, x) < R\}.$$

If  $X$  is a riemannian manifold, the closure of  $B_R(X, x)$  is homeomorphic to a closed ball at least for sufficiently small  $R$ . The injectivity radius of  $X$  at  $x \in X$  is the first supremum of such radii, and denoted by  $\text{inj}_x X$ . If  $X$  has a boundary  $\partial X$ , we choose the supremum by furthermore requiring that  $B_R(X, x)$  does not touch  $\partial X$ . The injectivity radius of  $X$  is the infimum of injectivity radii of the points in  $X$ . We denote it by

$$\text{inj } X := \inf \{\text{inj}_x X \mid x \in X\}.$$

The injectivity radius for manifolds with nonempty boundary by this definition would not be interesting since the points close to the boundary always make it vanishing.

Let  $C$  be an orientable hyperbolic 3-cone-manifold of finite volume with compact singularity. The singular set  $\Sigma$  is assumed to form a link

$$\Sigma = \Sigma^1 \cup \dots \cup \Sigma^n$$

of  $n$  components. To each component  $\Sigma^j$  of  $\Sigma$ , associated is a cone angle  $\alpha^j \in (0, \infty)$ . The angle set  $A$  of  $C$  is a vector

$$A = (\alpha^1, \dots, \alpha^n)$$

of cone angles.

$C$  carries a nonsingular but incomplete hyperbolic structure on the complement of the singularity

$$N = C - \Sigma.$$

$C$  itself inherits a metric induced from a Riemannian metric on  $N$ . We assume that  $C$  is complete with respect to this metric. In particular, the metric completion of  $N$  is identical to  $C$ . We have a developing map of  $N$  from its universal covering space  $\widetilde{N}$ ,

$$\mathcal{D}_C : \widetilde{N} \rightarrow \mathbf{H}^3,$$

and a holonomy representation

$$\rho_C : \Pi = \pi_1(N) \rightarrow \mathrm{PSL}_2(\mathbf{C}).$$

They are called a developing map and a holonomy representation of a cone-manifold  $C$ . A developing map is a local isometry, but never be injective. A holonomy representation is hardly discrete nor faithful.

Let  $m_j$ ,  $j = 1, 2, \dots, n$ , be an oriented meridional loop for each component of  $\Sigma$ . The image  $\rho_C(m_j)$  of a meridian  $m_j$  by the holonomy representation is an elliptic element rotating  $\mathbf{H}^3$  by  $\alpha^j$  about the axis, though the rotation angle of  $\rho_C(m_j)$  makes sense only modulo  $2\pi$ .

The injectivity radius of  $C$  at  $x \in N = C - \Sigma$  is to be the injectivity radius of  $N$  at  $x$  and denoted by

$$\mathrm{inj}_x C \, (:= \mathrm{inj}_x N).$$

The global injectivity radius of a cone-manifold  $C$  by this definition would not be interesting since if the singular set is nonempty, then the points close to the singularity always make it vanishing.

**Definition .** A *topological type* of a cone-manifold  $C$  is a homeomorphism type of a pair  $(C, \Sigma)$ . We say  $C$  is *homeomorphic* to  $C_1$  for short if there is a homeomorphism between  $(C, \Sigma)$  and  $(C_1, \Sigma_1)$ . More strong relation is an *isomorphism type*. Two cone-manifolds are *isomorphic* if they share not only topological types but also cone angles, more precisely if there is a homeomorphism between  $(C, \Sigma)$  and  $(C_1, \Sigma_1)$  so that the corresponding components of the singularity share the same cone angles. Such a homeomorphism is called an *isomorphism*. A self isomorphism is called an *automorphism* as usual. The strongest relation is an *isometry type* whose definition would be obvious.

**Remark .** The global rigidity is the claim that the isomorphism type and the isometry type are the same.

**1.2. Nonsingular Parts.** The following theorem was proved in [9] under an extra angle assumption using Hodgson-Kerckhoff's local rigidity. Here we present a quick argument due to Steve Kerckhoff which does not use the local rigidity and works without any angle assumption. As we mentioned in the introduction, Zhou also showed the following theorem in [19] in slightly different manner.

**Theorem 1.2.1.** *The underlying space of a nonsingular part  $N$  of an orientable hyperbolic 3-cone-manifold  $C$  of finite volume carries a complete negatively curved metric. In particular it is homeomorphic to an interior of a compact irreducible atoroidal 3-manifold with toral boundary which admits no Seifert fibrations. Moreover, it admits a complete hyperbolic structure of finite volume.*

*Proof.* In the cylindrical coordinates around each component of the singularity  $\Sigma$ , the metric has the form

$$d\delta^2 + \sinh^2 \delta d\theta^2 + \cosh^2 \delta d\lambda^2$$

where  $\delta$  is the distance from the singularity,  $\lambda$  is the distance along the singularity,  $\theta$  is the angular measure around the singularity. Choose  $\varepsilon > 0$  small enough so that an  $\varepsilon$ -tubular neighborhood of  $\Sigma$  is a disjoint union of an  $\varepsilon$ -tubular neighborhood of each component of  $\Sigma$ . Also choose monotone  $C^\infty$ -functions  $\varphi(\delta)$  and  $\psi(\delta)$  in terms of  $\delta$  so that

$$\varphi(\delta) = \begin{cases} 1 & \text{if } \delta \geq \varepsilon, \\ O(1/\delta) & \text{if } \delta \rightarrow 0, \end{cases}$$

and

$$\psi(\delta) = \begin{cases} \cosh \delta & \text{if } \delta \geq \varepsilon, \\ O(\delta) & \text{if } \delta \rightarrow 0, \end{cases}$$

where  $O(\cdot)$  is the Landau symbol, and modify the metric in an  $\varepsilon$ -tubular neighborhood of each component of  $\Sigma$  by

$$\varphi^2(\delta)d\delta^2 + \sinh^2 \delta d\theta^2 + \psi^2(\delta)d\lambda^2.$$

This gives a complete metric on the nonsingular part  $N = C - \Sigma$  since  $\varphi(\delta)$  diverges when  $\delta \rightarrow 0$ .

Let us compute the sectional curvature for this new metric. For notational convenience, we set  $\delta = x_1$ ,  $\theta = x_2$  and  $\lambda = x_3$ . By a computation of the Christoffel symbols, we have the evaluation of the connection with respect to this basis,

$\nabla_{\partial/\partial x_i}(\partial/\partial x_j)$	1	2	3
1	$\varphi'/\varphi \cdot \partial/\partial x_1$	$\coth \delta \cdot \partial/\partial x_2$	$\psi'/\psi \cdot \partial/\partial x_3$
2	$\coth \delta \cdot \partial/\partial x_2$	$-\sinh \delta \cosh \delta / \varphi^2 \cdot \partial/\partial x_1$	0
3	$\psi'/\psi \cdot \partial/\partial x_3$	0	$-\psi\psi'/\varphi^2 \cdot \partial/\partial x_1$

It is symmetric and hence the Riemannian curvature tensor

$$R(X, Y, Z) = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})(Z)$$

has been diagonalized, i.e.,  $\langle R(X, Y, Z), W \rangle = 0$  unless  $X \neq Y$  and either  $(X, Y) = (Z, W)$  or  $(X, Y) = (W, Z)$ . Thus the sectional curvatures

$$K(X, Y) = -\frac{\langle R(X, Y, X), Y \rangle}{\|X\|^2 \cdot \|Y\|^2 - \langle X, Y \rangle^2}$$

all are convex combinations of the three sectional curvatures  $K_{ij} = K(\partial/\partial x_i, \partial/\partial x_j)$  with  $\{i, j\} = \{1, 2, 3\}$ . By a direct computation, we have

$$\begin{aligned} K_{12} &= \frac{1}{\varphi^2} \left( \frac{\varphi' \coth \delta}{\varphi} - 1 \right), \\ K_{13} &= -\frac{1}{\varphi^2} \left( \frac{\psi''}{\psi} - \frac{\varphi' \psi'}{\varphi \psi} \right), \\ K_{23} &= -\frac{\psi' \coth \delta}{\varphi^2 \psi}. \end{aligned}$$

Then these three values stay negative away from zero for  $0 < \delta \leq \varepsilon$  by choosing  $\varphi$  and  $\psi$  appropriately, and that the volume change is still bounded. Hence the nonsingular part  $N$  admits a complete negatively curved metric of finite volume.

The topological properties in the statement all are now easily derived from the negativity of the new metric. Then by Thurston's uniformization theorem for Haken manifolds [10, 16],  $N$  admits a complete hyperbolic structure.  $\square$

It is convenient to adopt the convention that we regard a complete hyperbolic manifold with preferred meridional elements as a cone-manifold.

**Definition .** Let  $M$  be a complete orientable hyperbolic 3-cone-manifold of finite volume with prescribed meridional elements not only for the components of  $\Sigma$  but for some cusps. We call  $M$  also a cone-manifold. The *topological type* of  $M$  is a pair of the Dehn filled resultant of  $M$  along prescribed meridional curves for cusps in question and the union of  $\Sigma$  and surgery cores. The cone angles which should be assigned to the components for cusps all are zero.

**Notation .** If  $C$  is a compact orientable hyperbolic 3-cone-manifold in the original sense, then the nonsingular part together with the complete structure obtained by Theorem 1.2.1 is a cone-manifold in this new sense by assigning original meridional elements of  $C$ . We let  $C_{comp}$  be this particular cone-manifold without singularity. The topological types of  $C$  and  $C_{comp}$  are the same.

**Corollary 1.2.2.** *The group of automorphisms of an orientable hyperbolic 3-cone-manifold of finite volume up to isotopy is finite.*

*Proof.* Here we duplicate the proof in [9]. Any automorphism of  $C$  induces a homeomorphism of  $C_{comp}$  preserving meridians. By Mostow-Prasad rigidity, it is homotopic to an isometry of  $C_{comp}$ . Moreover by Waldhausen's theorem [17], this homotopy can be attained by an isotopy. The isotopy extends to an automorphism of  $C$ . Hence the group of automorphisms of  $C$  up to isotopy is realized as a subgroup of the isometry group of  $C_{comp}$ , which is finite.  $\square$

**Remark .** The existence of the singularity in the proof is needed to apply Waldhausen's result, and in fact the argument does not cover the case without singularity. This finiteness up to isotopy for nonsingular case had been a difficult question and was settled very recently by Gabai, Meyerhoff and Thurston [3] in full generality, though the finiteness up to homotopy is a consequence of Mostow rigidity.

**Corollary 1.2.3.** *An orientable hyperbolic 3-cone-manifold of finite volume admits only finitely many toral cusps which are foliated by horotori.*

*Proof.* Since the end of  $C_{comp}$  consists of finitely many toral ends, the end of  $C$  consists of finitely many components homeomorphic to the torus time  $\mathbf{R}$  at least topologically. To get a complete end, every element of a fundamental group of each end component must be mapped by the holonomy representation to a parabolic element with a common fixed point at the sphere at infinity  $\mathbf{S}_\infty^2$ . A small neighborhood of each component is then foliated by the quotient of horospheres by the holonomy image.  $\square$

We conclude this subsection by the following observation for loops in a hyperbolic 3-cone-manifold. Every loop in a hyperbolic manifold admits a length shortening free homotopy to either a closed geodesic (including a point) or an arbitrary short loop tending to the cusp. This will not be true for cone-manifolds in general, but we have its weaker version.

**Lemma 1.2.4.** *Any loop in the nonsingular part  $N$  of an orientable hyperbolic cone-manifold  $C$  of finite volume admits a length shortening homotopy to either a closed geodesic, an arbitrary short loop tending to the cusp or a piecewise geodesic loop hitting the singularity  $\Sigma$ .*

*Proof.* Let  $\ell$  be an arbitrary loop in the nonsingular part  $N$ . Fix a point  $p$  on  $\ell$ , and we straighten  $\ell$ . Then it defines a length shortening homotopy to at least a piecewise geodesic loop hitting the singularity  $\Sigma$ . If  $\Sigma$  does not obstruct the straightening, the length shortening homotopy reaches to a geodesic path  $\ell'$  based at  $p$ . We then homotop  $\ell'$  to a shorter geodesic path by sliding the reference point. This length shortening homotopy either terminates at some stage or tends to the cusp. When it terminates, either the homotopy hits  $\Sigma$  or it reaches to a closed geodesic.  $\square$

**Remark .** The proof is valid even when  $C$  is of infinite volume, or when the singularity is noncompact.

**1.3. Volumes.** The volume of a hyperbolic cone-manifold  $C$  can be expressed in terms of an integral over the canonical section of the associated flat  $\mathbf{H}^3$ -bundle over  $N$ . To say more precise, let  $\widetilde{N}$  be the universal covering space of  $N$ . Each element  $\iota \in \Pi = \pi_1(N)$  acts on  $\widetilde{N}$  by a deck transformation and on  $\mathbf{H}^3$  by  $\rho_C(\iota) \in \rho_C(\Pi)$ . Denote by  $E$  the fiber product  $\widetilde{N} \times \mathbf{H}^3 / (\Pi \times \rho_C)$ . It is a  $\mathbf{H}^3$ -bundle over  $N$  with a structure group in  $\mathrm{PSL}_2(\mathbf{C})$ . The developing map  $\mathcal{D}_C : \widetilde{N} \rightarrow \mathbf{H}^3$  defines a section  $id \times \mathcal{D}_C : \widetilde{N} \rightarrow \widetilde{N} \times \mathbf{H}^3$  which descends to a canonical one  $s_0 : N \rightarrow E$  with respect to the hyperbolic structure on  $N$ . The volume of  $C$  is then identified with the integral,

$$\mathrm{vol} C = \int_N s_0^* dv, \quad (1.1)$$

where  $dv$  is the volume form of  $\mathbf{H}^3$ .

**Lemma 1.3.1.** *Suppose  $C$  is compact. If a section  $s_1 : N \rightarrow E$  agrees with the canonical one  $s_0$  on a small tubular neighborhood of  $\Sigma$ , then*

$$\int_N s_1^* dv = \int_N s_0^* dv.$$

In particular, the volume defined in (1.1) depends only on the behavior of the section near the singularity.

*Proof.* Since  $\mathbf{H}^3$  is contractible, there is a homotopy  $s_t$  of sections connecting  $s_0$  and  $s_1$ . Let  $N_\varepsilon$  be a compact exterior of an  $\varepsilon$ -tubular neighborhood of  $\Sigma$ . Choose  $\varepsilon$  small enough so that  $s_0 = s_1$  on that neighborhood of  $\Sigma$ . Then by Stokes,

$$\begin{aligned} \int_N s_1^* dv - \int_N s_0^* dv &= \int_{N_\varepsilon} s_1^* dv - \int_{N_\varepsilon} s_0^* dv \\ &= \int_{\partial N_\varepsilon \times [0,1]} s_t^* dv \\ &= \int_{N_\varepsilon \times [0,1]} d(s_t^* dv). \end{aligned}$$

The last integral is zero.  $\square$

**Proposition 1.3.2.** *Given a compact orientable hyperbolic 3-cone-manifold  $C$  with the angle set  $A = (\alpha^1, \dots, \alpha^n)$ , there is a constant  $V_{max}$  so that if a hyperbolic cone-manifold  $C_1$  homeomorphic to  $C$  has an angle set  $A_1 = (\alpha_1^1, \dots, \alpha_1^n)$  with  $\alpha_1^j \leq \alpha^j$  for all  $1 \leq j \leq n$ , then  $\text{vol } C_1 \leq V_{max}$ .*

*Proof.* Take a fine geodesic triangulation  $K$  of  $C$  so that  $\Sigma$  is a subcomplex and a star neighborhood of  $\Sigma$  is a closed regular neighborhood. Then since  $\alpha_1^j \leq \alpha^j$  for all  $j$ , we may choose a homeomorphism  $\varphi : (C, \Sigma) \rightarrow (C_1, \Sigma_1)$  so that any 3-simplex in a regular neighborhood of  $\Sigma$  is mapped to an honest geodesic simplex in  $C_1$ .

Let  $\widetilde{K}^{(0)}$  be the preimage of the 0-skeleton  $K^{(0)}$  in the universal cover  $\widetilde{N}_1$ . Then the developing map  $\mathcal{D}_{C_1}$  defines the map  $id \times \mathcal{D}_{C_1}|_{\widetilde{K}^{(0)}} : \widetilde{K}^{(0)} \rightarrow \widetilde{N}_1 \times \mathbf{H}^3$  which extends to an equivariant continuous map of  $\widetilde{N}_1$  by straightening. Since it is equivariant, it descends to a section  $s_1 : N_1 \rightarrow E_1 = \widetilde{N}_1 \times \mathbf{H}^3 / (\Pi_1 \times \rho_{C_1})$ .  $s_1$  is identical to the canonical section  $s_0 : N_1 \rightarrow E_1$  near  $\Sigma_1$ . Hence by the previous lemma, we have

$$\text{vol } C_1 = \int_{N_1} s_1^* dv.$$

The right hand side is a total sum of the signed volumes of 3-simplices appeared by straightening. However since the volume of a 3-simplex is uniformly bounded by a constant  $v_3$  (= the volume of a regular ideal tetrahedra), this sum is bounded by  $V_{max} = kv_3$ , where  $k$  is the number of 3-simplices in  $K$ . The constant  $V_{max}$  obviously depends only on  $C$  and not on  $C_1$ .  $\square$

**Remark .** A number of tetrahedra needed to triangulate a neighborhood of the singularity  $\Sigma$  in  $C$  depends in fact on the cone angles. However it is bounded by some constant depending only on the upper bound of the cone angles and not on any particular  $C$ .

## 2. CONE ANGLES $\leq 2\pi$

In this section, we review two machineries to study deformations of a hyperbolic 3-cone-manifold  $C$  when cone angles are at most  $2\pi$ . One is the local rigidity by Hodgson and Kerckhoff [7] and the other is the pointed Hausdorff-Gromov topology studied in [5].



**2.1. Local Rigidity.** Let us recall what the deformation of a hyperbolic 3-cone-manifold  $C$  is.

**Definition .** A deformation of a hyperbolic 3-cone-manifold  $C$  is a hyperbolic 3-cone-manifold  $C_1$  together with a reference homeomorphism  $\xi_1 : (C, \Sigma) \rightarrow (C_1, \Sigma_1)$ . Two deformations  $(C_1, \xi_1)$  and  $(C_2, \xi_2)$  of  $C$  are equivalent if there is an isometry  $\psi : C_1 \rightarrow C_2$  so that  $\xi_2$  is isotopic to  $\psi \circ \xi_1$ . A composition of a developing map  $\mathcal{D}_{C_1}$  with a lift  $\tilde{\xi}_1$  of  $\xi_1$  defines a continuous map  $\mathcal{D}_{C_1} \circ \tilde{\xi}_1 : \tilde{N} \rightarrow \mathbf{H}^3$ . The developing map is well defined up to multiplication of isometries of  $\mathbf{H}^3$ , and hence a deformation can be considered as a point on a quotient space of the mapping space  $\mathcal{M}(\tilde{N}, \mathbf{H}^3)$  by  $\mathrm{PSL}_2(\mathbf{C})$ -action on the images. The quotient space carries a topology induced by a compact open topology on  $\mathcal{M}(\tilde{N}, \mathbf{H}^3)$ . The set of equivalence classes of deformations of  $C$  carries a further quotient topology by the action on the source of the group of lifts of automorphisms of  $C$  which are isotopic to the identity. We say  $(C_1, \xi_1)$  is a small deformation of  $(C, id)$  if it is close to  $C$  in this topology.

**Remark .** The reference homeomorphism for the deformation is to fix an isotopy class of the model. In fact, only its isotopy class is significant.

The topology on the set of equivalence classes of deformations of  $C$  turns out to be not quite complicated by the local rigidity. To see this, we review a few topological properties of the space of representations of the group  $\Pi = \pi_1(N)$  in  $\mathrm{PSL}_2(\mathbf{C})$ . The space of such representations carries a natural algebraic topology. There is a canonical projection to the set of conjugacy classes  $\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$ . A small neighborhood of a conjugacy class represented by a holonomy representation of a cone-manifold will be well behaved.

An orientation preserving isometry  $\varphi$  of  $\mathbf{H}^3$  can be represented by a matrix  $\Phi$  in  $\mathrm{SL}_2(\mathbf{C})$ . The complex length of  $\varphi$  is a twice of an appropriate branch of the log of an eigenvalue of  $\Phi$ . It measures how much  $\varphi$  translates  $\mathbf{H}^3$  with twist along an invariant geodesic. Since there are two choices to be made, we adopt the following convention. To each oriented meridional element  $m_j$ , we choose a complex length of  $\rho_C(m_j)$  by a multiple of its cone angle with the complex unit  $\sqrt{-1}$ , which we denote by  $\mathcal{L}_{m_j}(\rho_C)$ . It is equal to  $\alpha^j \sqrt{-1}$ . Then orient the invariant geodesic  $\ell_C^j$  of  $\rho_C(m_j)$  so that the rotational direction of  $m_j$  is counter clockwise. There are two ways to continuously extend this assignment of a complex length to that of transformation represented by  $\rho(m_j)$  where  $\rho$  is close to  $\rho_C$ . Since  $\rho$  is close to  $\rho_C$ , the invariant geodesic  $\ell^j$  of  $\rho(m_j)$  is also close to  $\ell_C^j$  and inherits an orientation. We choose the sign of the real part according to whether  $\rho(m_j)$  translates the point on  $\ell^j$  in positive or negative direction.

This convention extends to a continuous map, which we denote by  $\mathcal{L}_{m_j}$ , defined on the component  $\mathrm{Hom}_C(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$  of  $\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$  containing the conjugacy class of  $\rho_C$ . Arranging these maps, we obtain

$$\mathcal{L}_m : \mathrm{Hom}_C(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C}) \rightarrow \mathbf{C}^n,$$

where  $\mathcal{L}_m(\rho) = (\mathcal{L}_{m_1}(\rho), \dots, \mathcal{L}_{m_n}(\rho))$ .

**Theorem 2.1.1** (Hodgson-Kerckhoff's Local Rigidity [7]). *If the cone angles assigned to the components of  $\Sigma$  in  $C$  all are positive and  $\leq 2\pi$ , then  $\mathcal{L}_m$  is a local diffeomorphism near the conjugacy class represented by  $\rho_C$ .*

In particular, the complex dimension of  $\text{Hom}(\Pi, \text{PSL}_2(\mathbf{C}))/\text{PSL}_2(\mathbf{C})$  near the class represented by  $\rho_C$  is equal to  $n$ , the number of components of  $\Sigma$ , and  $\rho_C$  represents a smooth point.

Removing a small tubular neighborhood of  $\Sigma$  from  $C$ , we obtain a compact hyperbolic manifold with boundary. Its nearby deformations supported on all but a small neighborhood of the boundary are parameterized by holonomy representations [14]. The parameter in terms of complex lengths is more geometrically described in [7, 14]. For each value near  $\mathcal{L}_m(\rho_C)$ , there is a unique way to fill the boundary by the hyperbolic Dehn filling theory [14]. If the value of  $\mathcal{L}_m$  is in purely imaginal part  $\Theta = (\Im \mathbf{C})^n \subset \mathbf{C}^n$ , then the filled resultant is a hyperbolic cone-manifold with the same topological type but with perturbed cone angles. Hence the local rigidity together with the hyperbolic Dehn surgery filling imply the unique existence of a small deformation of  $C$  if the perturbation of cone angles is small enough. Namely the possible range of perturbation is open.

When we start with a noncompact 3-cone-manifold without singularity, such as  $C_{\text{comp}}$  then the existence of a small deformation is nothing but the conclusion of the hyperbolic Dehn filling theory for complete manifolds with a specified direction. In particular, there is a unique small deformation with perturbed cone angles for this case also. One can summarize the conclusion by

**Corollary 2.1.2.** *Let  $C$  be an orientable hyperbolic 3-cone-manifold of finite volume so that the cone angles assigned to the components of  $\Sigma$  all are at most  $2\pi$ , possibly some or all of them are zero. Then there is a unique small deformation of  $C$  with perturbed cone angles if the perturbed cone angles are close enough to the initial ones.*

**Remark .** A continuous path on the space of representations whose image by  $\mathcal{L}_m$  is contained in purely imaginal part does not always correspond to the deformations of a cone-manifold in the full range.

**2.2. Pointed Hausdorff-Gromov Topology.** We review the pointed geometric convergence of complete metric spaces due to Gromov in [5], which generalizes the idea of Hausdorff topology on the set of all compact subset in a complete metric space.

**Definition .** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed complete metric spaces. A relation  $R \subset X \times Y$  is an  $\varepsilon$ -approximation between  $(X, x_0)$  and  $(Y, y_0)$  if

1. there is  $y \in Y$  such that  $x_0 R y$  and  $d_Y(y_0, y) < \varepsilon$ ,
2. there is  $x \in X$  such that  $x R y_0$ , and  $d_X(x_0, x) < \varepsilon$ ,
3.  $pr_X(R \cap (B_{1/\varepsilon}(X, x_0) \times B_{1/\varepsilon}(Y, y_0))) = B_{1/\varepsilon}(X, x_0)$ ,
4.  $pr_Y(R \cap (B_{1/\varepsilon}(X, x_0) \times B_{1/\varepsilon}(Y, y_0))) = B_{1/\varepsilon}(Y, y_0)$ , and
5. for any  $x, x' \in B_{1/\varepsilon}(X, x_0)$  and  $y, y' \in B_{1/\varepsilon}(Y, y_0)$  with  $x R y$  and  $x' R y'$ , we have  $|d_X(x, x') - d_Y(y, y')| < \varepsilon$ .

**Definition .** A sequence  $\{(X_i, x_i)\}$  of pointed complete metric spaces is said to converge geometrically to  $(Y, y)$  if for any  $\varepsilon > 0$ , there exists  $i_0$  such that there is an  $\varepsilon$ -approximation between  $(X_i, x_i)$  and  $(Y, y)$  for all  $i > i_0$ .

An interesting class  $\mathcal{C}$  of pointed complete metric spaces with respect to this convergence is the one whose member  $(X, x)$  has the property;

$$\text{The closure of } B_R(X, x) \text{ is compact for all } R > 0. \quad (2.1)$$

It is shown in [5] that the geometrically convergent sequence  $\{(X_i, x_i)\}$  contained in the class  $\mathcal{C}$  geometrically converges to a unique complete metric space up to isometry. Hence the geometric convergence defines a Hausdorff topology on the set of isometry classes of  $\mathcal{C}$ . We call it the pointed Hausdorff-Gromov topology.

If furthermore we have some uniformity for the local structure of the metric spaces, then the set of isometry classes of such spaces together with the pointed Hausdorff-Gromov topology becomes precompact. In particular, any sequence in the space contains a subsequence converging geometrically to a unique metric space. The following criterion by Gromov is useful.

**Proposition 2.2.1** (Gromov [5]). *Let  $\{(X_i, x_i)\}$  be a sequence of pointed complete metric spaces with the property (2.1) for all  $i$ , then the sequence contains a geometrically convergent subsequence if and only if there is a subsequence  $\{k\} \subset \{i\}$  so that for any  $R, \varepsilon > 0$ ,*

$$\min \# \{ \varepsilon\text{-balls covering } B_R(X_k, x_k) \}$$

*is bounded by some constant depending only on  $R$  and  $\varepsilon$ , where the minimum is taken over all  $\varepsilon$ -coverings of  $B_R(X_k, x_k)$ .*

The class of metric spaces we are concerned with is quite restricted, but not only hyperbolic cone-manifolds. We will work also on euclidean cone-manifolds as their rescaling limits.

**Notation .** Fix a number  $L \leq -1$ , which will bound curvature.  $\mathcal{C}_{[L,0]}^\theta$  will be the set of pointed compact orientable cone-manifolds of constant sectional curvature  $= K$ , where  $K \in [L, 0]$ , so that the cone angles assigned to the components of the singularity all are at most  $\theta$ .  $\mathcal{C}_K^\theta$  is a subset of  $\mathcal{C}_{[L,0]}^\theta$  consisting of cone-manifolds with a particular curvature constant  $K$ .

**Remark .** Each member of  $\mathcal{C}_{[L,0]}^\theta$  is compact and hence has the property (2.1).

Choose  $C \in \mathcal{C}_K^{2\pi}$  and  $x \in C$  not lying on  $\Sigma$ . Here we bound cone angles by  $2\pi$ . The set of points in  $C$  which admit at least two shortest geodesic paths to  $x$  in  $C$  is called a cut locus of  $C$  with respect to the reference point  $x$ . The cut locus is a connected geodesic cell complex in  $C$ . The complement of the cut locus consists of shortest rays to  $x$ ,

$$P_x = \{y \in C \mid y \text{ admits the unique shortest path to } x\},$$

and is called a Dirichlet fundamental domain of  $C$  about  $x$ .

**Lemma 2.2.2.** *The Dirichlet fundamental domain  $P_x$  of  $C \in \mathcal{C}_K^{2\pi}$  about  $x$  is isometrically realized as an interior of a starshaped polyhedron in the simply connected 3-dimensional space  $\mathbf{H}_K$  of constant curvature  $= K$ . The closure is a starshaped polyhedron. Furthermore if  $C \in \mathcal{C}_K^\pi$ , that is to say, cone angles all are  $\leq \pi$ , then  $P_x$  is convex.*

*Proof.* Obvious.  $\square$

We call this embedded compactified polyhedron a Dirichlet polyhedron of  $C$  about  $x$ , and denote it again by  $P_x$ . Namely,  $P_x$  stands for an open dense subset in  $C$ , and simultaneously, a compact polyhedron in  $\mathbf{H}_K$ .

**Proposition 2.2.3.** *Fix a curvature bound  $L \leq -1$ . Any sequence  $\{C_i\} \subset \mathcal{C}_{[L,0]}^{2\pi}$  with reference points  $x_i \in C_i$  contains a subsequence converging geometrically to a complete metric space  $(C_*, x_*)$ .*

*Proof.* Let  $(C, x_0)$  be a pointed compact orientable 3-cone-manifold with constant sectional curvature  $= K$  where  $L \leq K \leq 0$ . If  $x_0 \notin \Sigma$ , then

$$\begin{aligned} \min \# \{ \varepsilon\text{-balls covering } B_R(C, x_0) \} &\leq \min \# \{ \varepsilon\text{-balls covering } B_R(P_{x_0}, x_0) \} \\ &\leq \min \# \{ \varepsilon\text{-balls covering } B_R(\mathbf{H}_K, x_0) \} \\ &\leq \min \# \{ \varepsilon\text{-balls covering } B_R(\mathbf{H}_L, x_0) \}, \end{aligned}$$

where the minimum is taken all over  $\varepsilon$ -ball coverings of the target. The last bound depends only on  $R$  and  $\varepsilon$ , and not on any particular  $C$  or  $K$ . If  $x_0 \in \Sigma$ , the same bound actually works by choosing the center of a Dirichlet polyhedron near  $x_0$  but not on  $\Sigma$ . Thus the result follows from Gromov's Criterion.  $\square$

**Definition .** We call  $(C_*, x_*)$  a geometric limit of  $\{(C_i, x_i)\}$ .

**Remark .** If  $C_*$  is compact, the isometry class of  $C_*$  does not depend on the choice of the reference points  $x_i \in C_i$  in the sequence.

**Remark .** The set of equivalence classes of deformations of an orientable hyperbolic 3-cone-manifold  $C$  with cone angle  $\leq 2\pi$  carries a topology well described by the local rigidity. Assigning the isometry class of a deformation to each deformation, we get a map to the set of isometry classes of metric spaces in  $\mathcal{C}_{-1}^{2\pi}$  together with the pointed Hausdorff-Gromov topology. It is not quite hard to see that this map is continuous.

### 3. CONE ANGLES $\leq \pi$

In this section, we discuss three relative constants for hyperbolic 3-cone-manifolds with cone angles  $\leq \pi$ , two of which dominate the local structure away from the singularity, and the other one of which is related with the geometry of a tubular neighborhood of the singularity. The angle condition “ $\leq \pi$ ” does not explicitly appear, but instead the fact that the Dirichlet polyhedron is convex, which is a conclusion of the angle condition, will be used often. We also discuss how cusp opening deformations occur locally.

**3.1. Thin Parts.** The constant in the following lemma is to claim that the injectivity radius decreases uniformly, like in the hyperbolic manifolds, away from the singularity.

**Lemma 3.1.1.** *Fix a curvature bound  $L \leq -1$ . Given positive numbers  $D, I, R > 0$ , there is a constant  $U(D, I, R, L) > 0$  such that if  $C \in \mathcal{C}_{[L,0]}^\pi$ ,  $x \in C$  with  $d(x, \Sigma) \geq D$  and  $\text{inj}_x C \geq I$ , then*

$$\text{inj}_y C \geq U(D, I, R, L)$$

*for any  $y \in C$  with  $d(y, \Sigma) \geq D$  and  $d(y, x) \leq R$ .*

*Proof.* Assume that there are no such uniform bounds. Then there is a sequence of cone-manifolds  $\{C_i\} \subset C_{[L,0]}^\pi$  and points  $x_i, y_i \in C_i$  such that

1.  $d(x_i, \Sigma_i), d(y_i, \Sigma_i) \geq D$ ,
2.  $\text{inj}_{x_i} C_i \geq I$ ,
3.  $d(y_i, x_i) \leq R$  and
4.  $\text{inj}_{y_i} C_i \leq 1/i$ .

Take a Dirichlet polyhedron  $P_{y_i}$  of  $C_i$  about  $y_i$  in  $\mathbf{H}_{K_i}$ , where  $K_i$  is a curvature constant of  $C_i$ . There are points  $p_i, q_i$  on  $\partial P_{y_i}$  which are identified in  $C_i$  and attain the shortest distance to  $y_i$  from  $\partial P_{y_i}$ . The union of these paths forms a homotopically nontrivial shortest loop  $\ell_i$  in  $C_i$  based at  $y_i$ .

$p_i$  and  $q_i$  are on the interior of the faces of  $P_{y_i}$  respectively. Since the cone angles are  $\leq \pi$  and hence the Dirichlet polyhedron  $P_{y_i}$  is convex,  $P_{y_i}$  is bounded by the extension of two faces which support  $p_i$  and  $q_i$ . If the faces tend to be parallel as  $i \rightarrow \infty$ , then the volume of  $B_{R+I}(C_i, y_i)$  approaches zero. This is a contradiction since it must contain the ball  $B_I(C_i, x_i)$  whose volume admits nonzero lower bound by (2).

If not,  $\ell_i$  meets at  $y_i$  with angle uniformly away from  $\pi$ . Let us lift  $\ell_i$  to a geodesic segment  $s_i$  in  $\mathbf{H}_{K_i}$  based at  $y_i$  such that  $p_i$  is the middle point. Then  $\rho_i(\ell_i)$  acts  $\mathbf{H}_{K_i}$  by either a loxodromic (translation with twist) motion or an elliptic rotation. In both cases, the orbit of  $s_i$  by the action of a group generated by  $\rho_i(\ell_i)$  forms a piecewise geodesic immersed line rounding around the axis of  $\rho_i(\ell_i)$ . Since the corner of this line at the orbit of  $y_i$  has an angle uniformly away from  $\pi$  with respect to  $i$ , and since length of  $s_i$ , which equals the length of  $\ell_i$ , approaches zero when  $i \rightarrow \infty$ , the immersed line must squeeze onto or into the axis of  $\rho_i(\ell_i)$  according to whether  $\rho_i(\ell_i)$  is loxodromic or elliptic. In particular in both cases, the axis of  $\rho_i(\ell_i)$  becomes close to  $y_i$  when  $i \rightarrow \infty$ .

If there is a subsequence  $\{k\} \subset \{i\}$  such that  $\rho_k(\ell_k)$  all are loxodromic, then the translation distance becomes also short. The path joining  $p_k$  and  $q_k$  in  $\mathbf{H}_{K_k}$  is equivariant tiny homotopic to the unit translation segment on the axis of  $\rho_k(\ell_k)$ . This equivariant homotopy must induce a tiny homotopy in  $C_k$  because of (1). Hence we obtain a very short closed geodesic in  $C_k$  near  $y_k$ . Then if we choose a new reference point  $z_k$  on this geodesic, the Dirichlet polyhedron about  $z_k$  will be bounded by almost parallel faces. This is a contradiction as before.

In the other case,  $\rho_k(\ell_k)$  all but finitely many exceptions are elliptic. Then the axis of an elliptic element comes close to  $y_k$ . Thus the length shortening tiny homotopy of  $\ell_k$  must hit the singularity by Lemma 1.2.4, though the hit singularity may not be the axis of  $\rho_k(\ell_k)$ . This contradicts (1).  $\square$

**3.2. Local Margulis.** The Margulis lemma for hyperbolic manifolds states that there is a universal constant depending only on the dimension which dominates the geometry and topology of thin parts. The cone-manifold admits in fact no such universal constant, however we may expect its relative version away from the singularity. The next lemma establishes that there is a Margulis like constant to control the geometry and topology of not absolute but relatively thin part with respect to the injectivity radius. We call it a local Margulis constant.

**Lemma 3.2.1.** *Given positive numbers  $D, R > 0$ , there is a constant  $V(D, R)$  such that if  $C \in \mathcal{C}_{-1}^\pi$ ,  $d(x, \Sigma) \geq D$  and  $\text{inj}_x C \leq V(D, R)$ , then  $(B_{R \cdot \text{inj}_x C}(C, x), x)$  is homeomorphic to  $(B_R(E, e), e)$  for some noncompact euclidean manifold  $E$  with  $\text{inj}_e E = 1$ .*

*Proof.* Assume that the conclusion is not true. Then there is a sequence of cone-manifolds  $\{C_i\} \subset \mathcal{C}_{-1}^\pi$  and points  $x_i \in C_i$  such that

1.  $d(x_i, \Sigma_i) \geq D$  and
2.  $\text{inj}_{x_i} C_i \leq 1/i$ , but
3.  $(B_{R \cdot \text{inj}_{x_i} C_i}(C_i, x_i))$  never be homeomorphic to  $(B_R(E, y), y)$  for some euclidean manifold with  $\text{inj}_y E = 1$ .

Then

$$R \leq iD \leq D / \text{inj}_{x_i} C_i$$

for  $i$  large enough, and we have

$$R \cdot \text{inj}_{x_i} C_i \leq D$$

for sufficiently large  $i$ . Hence  $(B_{R \cdot \text{inj}_{x_i} C_i}(C_i, x_i), x_i)$  is a subset of  $(B_D(C_i, x_i), x_i)$  for sufficiently large  $i$ . Notice that  $B_D(C_i, x_i)$  is nonsingular, so is  $B_{R \cdot \text{inj}_{x_i} C_i}(C_i, x_i)$ .

Multiplying  $1/\text{inj}_{x_i} C_i$  on the metric of  $C_i$ , we obtain a cone-manifold  $\overline{C}_i$  of constant curvature  $= -(\text{inj}_{x_i} C_i)^2 \geq -1$  such that  $\text{inj}_{\overline{x}_i} \overline{C}_i = 1$ . Hence we have a sequence of compact orientable cone-manifolds  $\{\overline{C}_i\}$  in  $\mathcal{C}_{[-1, 0]}^\pi$ . Then by Proposition 2.2.3, there is a subsequence  $\{k\} \subset \{i\}$  so that  $\{(\overline{C}_k, \overline{x}_k)\}$  converges geometrically to a complete metric space  $(\overline{C}_*, \overline{x}_*)$ .

The limit  $\overline{x}_*$  of reference points  $\{\overline{x}_k\}$  admits a neighborhood which is a limit of balls of radius 1 whose curvature tend to zero. On the other hand, an euclidean ball of radius 1 could be a geometric limit of this sequence. Hence by the uniqueness of the geometric limit,  $\overline{x}_*$  admits an euclidean ball neighborhood. This point will be a reference point for the other part.

To see a neighbor structure of the other part of  $\overline{C}_*$ , fix a constant  $R_1 > 0$  and choose any  $\overline{y}_* \in \overline{C}_*$  with  $d(\overline{y}_*, \overline{x}_*) \leq R_1$ , then it is a limit of points  $\{\overline{y}_k \in \overline{C}_k\}$  with, say,  $d(\overline{y}_k, \overline{x}_k) \leq 2R_1$ . By rescaling, we have

$$d(\overline{x}_k, \overline{\Sigma}_k) \geq D / \text{inj}_{x_k} C_k \geq kD,$$

and moreover,

$$d(\overline{y}_k, \overline{\Sigma}_k) \geq kD - 2R_1 \geq D$$

for sufficiently large  $k$ . Then by Lemma 3.1.1,  $\overline{y}_k$  admits a ball neighborhood of radius  $\geq U(D, 1, 2R_1, -1)$ . This radius bound does not depend on  $k$ , and hence a point  $\overline{y}_*$  admits a neighborhood which is a limit of balls of uniformly bounded radius whose curvature tend to zero. Thus  $\overline{x}_*$  admits an euclidean ball neighborhood. Now, since  $R_1 > 0$  was arbitrary, a point with an arbitrary long distance from  $\overline{x}_*$  admits an euclidean ball neighborhood. This shows that  $\overline{C}_*$  is an euclidean manifold without singularity. Moreover  $\text{inj}_{\overline{x}_*} \overline{C}_* = 1$  and  $\overline{C}_*$  is certainly noncompact.

Letting  $(E, e) = (\overline{C}_*, \overline{x}_*)$ , we will see that  $E$  has the property in the claim. Triangulate  $E$  by geodesic tetrahedra of uniform size and shape at least in a large compact set. Since it is a geometric limit, we may choose an approximate map  $\varphi_k$  from the 0-skeleton of a large compact set of  $E$  containing  $B_{D/\text{inj}_{x_k} C_k}(E, e)$  to a large compact set

of  $\overline{C}_k$  containing  $B_{D/\text{inj}_{x_k} C_k}(\overline{C}_k, \overline{x}_k)$  for sufficiently large  $k$ , where 4 vertices spanning an oriented simplex in  $E$  are mapped to 4-vertices spanning a simplex with the same orientation in the image. Then  $\varphi_k$  admits an obvious piecewise linear extension, which we denote again by  $\varphi_k$ , over a large compact set of  $E$ .  $\varphi_k$  might be locally branched along edges or vertices, however it will be a homeomorphism for further sufficiently large  $k$ , because otherwise,  $\varphi_k$ 's would not be accurate approximations. In particular, the restriction of  $\varphi_k$  for sufficiently large  $k$  induces a homeomorphism of  $(B_R(E, e), e)$  to  $(B_R(\overline{C}_k, \overline{x}_k), \overline{x}_k)$  after some tiny smoothing and hence to  $(B_{R \cdot \text{inj}_{x_k} C_k}(C_k, x_k), x_k)$ . This is a contradiction.  $\square$

**Remark .** A homeomorphism between  $(B_{R \cdot \text{inj}_x C}(C, x), x)$  and  $(B_R(E, e), e)$  can be chosen by the composition of an approximation and a rescaling, which is an almost equi-expansive map centered at  $x$ .

To see more topological structures of  $B_R(E, e)$ , recall that a noncompact euclidean manifold is a quotient of the euclidean space  $\mathbf{E}^3$  by a lattice  $\Gamma$  in  $\text{Isom}_+ \mathbf{E}^3$ .  $\Gamma$  is isomorphic to either  $\{0\}$ ,  $\mathbf{Z}$ ,  $\mathbf{Z} \oplus \mathbf{Z}$  or the fundamental group of the Klein bottle. In particular, any two elements in  $\Gamma$  are either mutually commute or anti-commute.

**Lemma 3.2.2.** *Let  $E$  be a noncompact euclidean manifold with a reference point  $e \in E$  such that  $\text{inj}_e E = 1$ . Then for any  $R > 0$ , the image of  $\pi_1(B_R(E, e), e) \rightarrow \pi_1(B_{2R}(E, e), e)$  induced by the inclusion is virtually abelian.*

*Proof.*  $\pi_1(B_R(E, e), e)$  is generated by geodesic loops of length  $\leq 2R$  which are not smooth only at the reference point  $e$ . Choose any two such loops  $\ell_1, \ell_2$ . Since they are commute or anti-commute in  $\pi_1(E)$ , the lift of a loop represented by  $\ell_1 \ell_2 \ell_1^{-1} \ell_2^\varepsilon$ , where  $\varepsilon = 1$  or  $-1$ , encloses a rectangular knot in  $\mathbf{E}^3$  where the edge length is at most  $2R$ .

The vertices are four lifts of the base point  $e$ . The preimage of  $B_{2R}(E, e)$ , which has twice in radius, contains a union of four balls of radius  $2R$  whose center lie on the vertices of the knot. Then the knot bounds a disk in the union of these four balls and hence  $\ell_1 \ell_2 \ell_1^{-1} \ell_2^\varepsilon$  becomes homotopic to zero in  $B_{2R}(E, e)$ . This shows that the image is generated by finitely many  $\ell$ 's where the generators are mutually commute or anti-commute. Such a group is virtually abelian.  $\square$

**3.3. Geometry of Tubes.** An abstract model for an equidistant tubular neighborhood of a singular geodesic in a cone-manifold will be useful for estimating several quantities. We call it a tube and discuss its geometry.

**Notation .**  $T_{\sigma, \delta, \theta, \tau}$  will be an equidistant tubular neighborhood of a singular component with radius  $\delta$  in a hyperbolic 3-cone-manifold where the length of a singular axis  $= \sigma$ , the cone angle  $= \theta$  and the twisting factor  $= \tau$ .

These four parameters determine the isometry class of  $T_{\sigma, \delta, \theta, \tau}$ . The boundary  $\partial T_{\sigma, \delta, \theta, \tau}$  carries an induced euclidean structure. A canonical rectangular fundamental domain of  $\partial T_{\sigma, \delta, \theta, \tau}$  by the meridional direction and its vertical direction has magnitude

$$\theta \sinh \delta \times \sigma \cosh \delta. \quad (3.1)$$

The surface area and volume of a tube depend only on the first three parameters,

$$\text{area } \partial T_{\sigma,\delta,\theta,\tau} = \theta \sigma \sinh \delta \cosh \delta, \quad (3.2)$$

$$\text{vol } T_{\sigma,\delta,\theta,\tau} = \frac{1}{2} \theta \sigma \sinh^2 \delta. \quad (3.3)$$

In particular,

$$\frac{\text{area } \partial T_{\sigma,\delta,\theta,\tau}}{\text{vol } T_{\sigma,\delta,\theta,\tau}} = 2 \coth \delta. \quad (3.4)$$

The twisting factor  $\tau$  does not affect the surface area and volume in fact, but does affect the euclidean structure of the boundary. Since the gluing of the right and left edges has no twisting factor, we have by (3.1)

$$\text{inj } \partial T_{\sigma,\delta,\theta,\tau} \leq \theta \sinh \delta. \quad (3.5)$$

The second factor “ $\sigma \cosh \delta$ ” of (3.1) does not say much about injectivity radii.

We will derive two implications from these quantities. They are about rank 2 cusp opening deformations of a tube, and the comparison of intrinsic and extrinsic injectivity radii of a point on the boundary of a tube embedded in a cone-manifold.

First of all, consider a sequence  $\{T_i\}$  of tubes which converges geometrically to a rank 2 cusp neighborhood by taking reference points on the boundaries. There are essentially two different ways. The simplest one in one way can be seen by setting

$$\begin{aligned} \theta_i &= 1 / \sinh \delta_i, \\ \sigma_i &= 1 / \cosh \delta_i \end{aligned}$$

and making  $\theta_i \rightarrow 0$  when  $i \rightarrow \infty$ . Then  $\delta_i \rightarrow \infty$  and  $\sigma_i \rightarrow 0$ . There are an elliptic and a loxodromic elements in  $\text{PSL}_2(\mathbf{C})$  which generate  $T_i$  for each  $i$ . In this deformations, they both approach parabolic elements which generate a rank 2 cusp when  $i \rightarrow \infty$ .

The other way involving a twisting factor  $\tau$  was discussed in [14]. To see one simple example, let  $\theta$  be a fixed positive constant and set

$$\sigma_i \sinh \delta_i \cosh \delta_i = 1 / \theta.$$

Choose  $\tau_i$  so that  $\{\partial T_i\}$  stays in a compact set in the moduli space of euclidean tori, we obtain a cusp opening family. In this case,  $\delta_i \rightarrow \infty$  and  $\sigma_i \rightarrow 0$  also. However, the elliptic elements associated to  $T_i$ 's diverge in  $\text{PSL}_2(\mathbf{C})$ , and a group generated by a loxodromic element converges geometrically (but not algebraically) to a rank 2 parabolic subgroup generating a cusp.

**Lemma 3.3.1.** *Let  $\{T_i\}$  be a sequence of tubes which converges geometrically to a rank 2 cusp neighborhood. Suppose that  $\text{area } \partial T_i$  is constant and that  $\{\partial T_i\}$  stays in a compact set in the moduli space of euclidean tori. If  $\theta_i > 0$  is bounded away from zero, then the lengths of curves on  $\partial T_i$ 's bounding a disk in  $T_i$  diverge as  $i \rightarrow \infty$ .*

*Proof.* Since the boundaries have bounded geometry and  $T_i$  approaches a rank 2 cusp neighborhood,  $\delta_i \rightarrow \infty$ . Then since the length of a curve bounding a disk in  $T_i$  is  $\theta_i \sinh \delta_i$ , it must diverge if  $\theta_i$  is bounded away from zero.  $\square$

Secondly, consider a tube  $T_{\sigma,\delta,\theta,\tau}^K$  with constant sectional curvature  $= K$ , where  $K$  lies in  $[L, 0]$  and is not necessarily  $-1$ . The modified formulas of the above ones for



$T_{\sigma,\delta,\theta,\tau}^K$  can be established according to the value of  $K$ , however its limiting behaviors caused by the limiting behavior of  $\sigma, \delta, \theta$  are essentially the same.

Regarding a tube as a model of an equidistant tubular neighborhood  $\mathcal{T} \subset C$  of some component of  $\Sigma$ , we compare the injectivity radius of  $\partial\mathcal{T}$  and injectivity radius of  $C$  at a point on  $\partial\mathcal{T}$ . Notice that since  $\partial\mathcal{T}$  is an euclidean torus,  $\text{inj}_x \partial\mathcal{T}$  does not depend on the choice of  $x \in \partial\mathcal{T}$  and is equal to  $\text{inj} \partial\mathcal{T}$ .

**Lemma 3.3.2.** *Fix a curvature bound  $L \leq -1$ . Given positive numbers  $D, I > 0$ , there is a constant  $W(D, I, L)$  such that if  $C \in \mathcal{C}_{[L,0]}^\pi$ ,  $\mathcal{T} \subset C$  is an equidistant tubular neighborhood of a component of  $\Sigma$  with radius  $\mathcal{T} \geq D$  and  $\text{inj} \partial\mathcal{T} \leq I$ , then*

$$(\text{inj}_w C \leq) \text{inj} \partial\mathcal{T} \leq W(D, I, L) \cdot \text{inj}_w C$$

for any  $w \in \partial\mathcal{T}$ , where the inequality in ( ) on the left hand side is obvious.

*Proof.* Let  $K'$  be a curvature constant of  $C$ . Multiplying  $1/\text{inj} \partial\mathcal{T}$  on the metric of  $C$ , we obtain  $\overline{C}$  of constant curvature  $K = K'(\text{inj} \partial\mathcal{T})^2 \geq I^2 L$ . Then  $\mathcal{T}$  becomes a tubular neighborhood  $\overline{\mathcal{T}}$  with radius  $\overline{\mathcal{T}} \geq D/I$  and also  $\text{inj} \partial\overline{\mathcal{T}} = 1$ .

Let  $T_{\sigma,\delta,\theta,\tau}^K$  be a tube isometric to  $\overline{\mathcal{T}}$ , where  $\delta = \text{radius} \overline{\mathcal{T}} \geq D/I$ , and consider a natural inclusion of tubes

$$T_{\sigma,\delta-D/2I,\theta,\tau}^K \subset T_{\sigma,\delta,\theta,\tau}^K.$$

Notice that the radius of the included tube is  $\geq D/2I$ .  $T_{\sigma,\delta,\theta,\tau}^K$  is a Riemannian manifold with boundary, and the quantity  $\text{inj}_x T_{\sigma,\delta,\theta,\tau}^K$  for  $x \in \partial T_{\sigma,\delta-D/2I,\theta,\tau}^K$  can be considered as a positive periodic function in terms of  $\tau$ , so that we have the minimum for fixed  $\sigma, \delta, \theta, K$ .

$$J_{\sigma,\delta,\theta}^K(D/I) = \min_{\tau} \text{inj}_x T_{\sigma,\delta,\theta,\tau}^K.$$

Then  $J_{\sigma,\delta,\theta}^K(D/I)$  becomes a function of  $\sigma > 0, \delta \geq D/I, \theta > 0$ , and  $0 \geq K \geq I^2 L$ . Since it approaches zero only when  $\text{inj} \partial T_{\sigma,\delta,\theta,\tau}^K \rightarrow 0$ , it attains the positive minimum  $J(D, I, L)$  when the variables run so that  $\text{inj} \partial T_{\sigma,\delta,\theta,\tau}^K = 1$ . In particular,  $\text{inj}_{\overline{x}} \overline{\mathcal{T}} \geq J(D, I, L)$  for any  $\overline{x} \in \overline{\mathcal{T}}$  with  $d(\overline{x}, \partial\overline{\mathcal{T}}) = D/2I$ .

Set a constant  $U$  by  $U(D/2I, J(D, I, L), D/2I, I^2 L)$  in Lemma 3.1.1, and choose any  $\overline{w} \in \partial\overline{\mathcal{T}}$ . Then there is the unique nearest point  $\overline{x}$  to  $\overline{w}$  in  $\overline{\mathcal{T}}$  so that  $d(\overline{x}, \overline{w}) = d(\overline{x}, \partial\overline{\mathcal{T}}) = D/2I$ . Since  $d(\overline{x}, \overline{\Sigma}) \geq D/2I$ ,  $\text{inj}_{\overline{x}} \overline{C} \geq \text{inj}_{\overline{x}} \overline{\mathcal{T}} \geq J(D, I, L)$  and  $d(\overline{w}, \overline{\Sigma}) \geq D/2I$ , we have a bound  $U \leq \text{inj}_{\overline{w}} \overline{C}$  by Lemma 3.1.1. Multiplying  $\text{inj} \partial\mathcal{T}$  on the metric, we obtain

$$U \cdot \text{inj} \partial\mathcal{T} \leq \text{inj}_w C.$$

The proof is done by letting  $W = 1/U$ . □

#### 4. CHOOSING GEOMETRICALLY CONVERGENT SEQUENCES

In this section, we choose a geometrically convergent sequence of deformations of a compact orientable hyperbolic 3-cone-manifold  $C$  with singularity  $\Sigma$ . This is to see what happens in the limit in the next two sections.

**4.1. Maximal Tubes.** We will often split the discussions in the later section according to whether the reference point is close to the singularity or not. For this, it is convenient to introduce the maximal tubular neighborhood of the singularity.

**Definition .** The maximal tube  $\mathcal{T}$  about  $\Sigma \subset C$  is a union of open tubular neighborhoods  $\mathcal{T}^j$ 's such that

1. each component  $\mathcal{T}^j \subset \mathcal{T}$  is an equidistant tubular neighborhood to the  $j$ th component  $\Sigma^j \subset \Sigma$  of the singularity,
2. among ones having the property (1), the set of radii arranged in order of magnitude from the smallest one is maximal in lexicographical order.

By the second condition, the maximal tube  $\mathcal{T}$  about  $\Sigma$  is uniquely determined. In fact,  $\mathcal{T}$  can be constructed as follows. Since the components of  $\Sigma$  are isolated, a  $\delta$ -tubular neighborhood of  $\Sigma$  is a union of disjoint tubular neighborhoods of  $\Sigma^j$ 's if  $\delta$  is very small. Thicken it gradually. Then some of components contact at a particular moment. Stop the growth of the components involved in contacting, and thicken the others furthermore. We will have the second contact moment. Do the same again. Continue this process up to the terminal moment, and we finally obtain the maximal tube  $\mathcal{T}$ .

Denote by  $\partial\mathcal{T}^j$  an abstract boundary of  $\mathcal{T}^j$ . It is an euclidean torus. The actual boundary  $\partial\mathcal{T}$  of  $\mathcal{T}$  in  $C$  is a union of isometrically embedded tori with a finite number of contact points.

**4.2. Geometrically Convergent Sequences.** Recall that  $A = (\alpha^1, \dots, \alpha^n)$  is an angle set of a compact orientable hyperbolic 3-cone-manifold  $C$ , where  $\alpha^j \leq \pi$  for all  $1 \leq j \leq n$ . Choose a sequence of deformations  $(C_i, \xi_i)$  of  $C$  such that

1.  $\alpha_i^j \leq \pi$  for any  $i$ , and
2.  $\lim_{i \rightarrow \infty} \alpha_i^j$  exists and equals  $\beta^j$ .

To see what happens when  $i \rightarrow \infty$ , we follow Thurston's strategy in [13] in the next two sections, that is, to analyze its possible geometric limit using hyperbolic geometry and 3-manifold topology.

Let  $c_i$  be the first contact point on  $\partial\mathcal{T}_i$ , that is to say, the point which admits two shortest path to  $\Sigma_i$  from  $\partial\mathcal{T}_i$ . Then by Proposition 2.2.3, there is a subsequence  $\{k\} \subset \{i\}$  so that  $\{(C_k, c_k)\}$  converges geometrically to a complete metric space  $(C_*, c_*)$ . We arrange more. Let  $f_k$  be the finest point on  $\partial\mathcal{T}_k$ , that is to say, the point on  $\partial\mathcal{T}_k$  which attains the minimum among  $\{\text{inj}_x C_k | x \in \partial\mathcal{T}_k\}$ . By choosing a further subsequence  $\{C_k\}$  with the same letter, we may assume that  $c_k$  and  $f_k$  stay on the components of  $\partial\mathcal{T}_k$  with constant reference numbers respectively. Namely,  $c_k$  always lies in  $c$ th component  $\partial\mathcal{T}_k^c$  for any  $k$ , and  $f_k$  does in  $\partial\mathcal{T}_k^f$ , where  $c$  and  $f$  here represent the reference numbers for the components.

The properties of the sequence so chosen is summarized in 4.3 below. In fact, we only use the properties listed in 4.3 in the next two sections.

**4.3. Properties.** A sequence of compact orientable hyperbolic 3-cone-manifold  $\{C_k\}$  with the angle set  $A_k = (\alpha_k^1, \dots, \alpha_k^n)$  has the properties:

1. each  $C_k$  is a deformation of  $C$  with a reference homeomorphism  $\xi_k : C \rightarrow C_k$ ,
2.  $\{(C_k, c_k)\}$  converges geometrically to  $(C_*, c_*)$  when  $k \rightarrow \infty$ ,

3.  $\alpha_k^j \leq \pi$  for all  $1 \leq j \leq n$  and any  $k$ ,
4.  $\alpha_k^j \rightarrow \beta^j$  when  $k \rightarrow \infty$ ,
5. the first contact point  $c_k$  lies on a component  $\partial\mathcal{T}_k^c$  with a constant reference number  $c$ ,
6. the finest points  $f_k$  lies on a component  $\partial\mathcal{T}_k^f$  with a constant reference number  $f$  and
7. there is a constant  $V_{max}$  such that  $\text{vol } C_k \leq V_{max}$ .

**Remark .** The sequence above is assumed to have only a geometric limit and the algebraic convergence with respect to the identification by  $\xi_k$  is not guaranteed. For instance, let  $C$  be a hyperbolic surface with homotopically nontrivial automorphism  $\varphi : C \rightarrow C$  which cannot be realized by an isometry, and define  $\xi_k : C \rightarrow C_k = C$  by

$$\xi_k = \begin{cases} id, & \text{if } k \text{ odd,} \\ \varphi, & \text{if } k \text{ even.} \end{cases}$$

Then the sequence  $\{C_k\}$  converges geometrically to  $C$  but the a sequence of holonomy representations  $\{\rho_k\}$  does not converge algebraically.

**Remark .** The property (7) is not a direct consequence of Proposition 1.3.2, however the bound above can be obtained by modifying its proof only a little because of the property (3). See also the remark after Proposition 1.3.2.

## 5. THICK TUBE

In this section, we study what happens to a geometric limit  $C_*$  of a sequence  $\{C_k\}$  of the deformations of  $C$  in 4.3 when maximal tubes of the singularity are uniformly thick.

**5.1. Brief Outline.** The underlying assumption throughout this section is

**Assumption 1.** *There is a constant  $D_1 > 0$  such that*

$$D_1 \leq \text{radius } \mathcal{T}_k^j$$

*for any  $1 \leq j \leq n$  and any  $k$ .*

Under the Assumption 1 above, we prove the following propositions in this section.

**Proposition 5.1.1.** *Under the Assumption 1, there is a constant  $I_1 > 0$  such that*

$$I_1 \leq \text{inj}_{f_k} C_k.$$

*for any  $k$ .*

The conclusion of this proposition is equivalent to the Assumption 1 since  $\text{inj}_{f_k} C_k \leq \min_j \text{radius } \mathcal{T}_k^j$ . The proof involves analysis of the local structure of cone-manifolds away from the singularity.

Using the conclusion of Proposition 5.1.1, we show

**Proposition 5.1.2.** *Under the conclusion of Proposition 5.1.1,  $C_*$  is a hyperbolic 3-cone-manifold of finite volume homeomorphic to  $C$ , where some components of  $\Sigma$  possibly disappear and create cusps.*

**Proposition 5.1.3.** *Under the conclusion of Proposition 5.1.1, a sequence  $\{\rho_k\}$  of holonomy representations of  $\{C_k\}$  contains a subsequence converging algebraically to the holonomy representation  $\rho_*$  of  $C_*$  with respect to the identification by  $\xi_k$ .*

**Definition .** A sequence  $\{C_k\}$  of deformations is said to converge strongly if it converges geometrically to a cone-manifold  $C_*$  homeomorphic to  $C$  and a sequence  $\{\rho_k\}$  of their holonomy representations converges algebraically to  $\rho_*$  with respect to the identification by  $\xi_k$ .

This definition is compatible for the existing one for discrete groups.

**Corollary 5.1.4.** *Under the Assumption 1,  $\{C_k\}$  contains a subsequence which converges strongly to a hyperbolic 3-cone-manifold  $C_*$  homeomorphic to  $C$ . If  $\beta^j > 0$  for all  $1 \leq j \leq n$ , then  $C_*$  is compact.*

*Proof.* This is a direct consequence of three propositions above. Suppose that the Assumption 1 is the case. By Proposition 5.1.1, the injectivity radius of the first contact points of maximal tubes are uniformly bounded away from zero. Then by Proposition 5.1.2, 5.1.3,  $C_*$  is a strong limit of  $\{C_k\}$  after taking a subsequence. In particular, the angle set of  $C_*$  is equal to  $B = (\beta^1, \dots, \beta^n)$ . If  $\beta^j > 0$  for all  $1 \leq j \leq n$ , then  $C_*$  admits no ends and hence is compact.  $\square$

**5.2. Boundary of Tubes.** Supposing that the Assumption 1 is the case throughout this subsection, we prove Proposition 5.1.1

**Lemma 5.2.1.** *There is a constant  $I_2$  such that*

$$\text{area } \partial \mathcal{T}_k^j \leq I_2,$$

*for any  $1 \leq j \leq n$  and any  $k$ .*

*Proof.* By the comparison (3.4) of the volume and the surface area of a tube, and by the volume bound (7) in 4.3, we have

$$\text{area } \partial \mathcal{T}_k^j = 2 \coth \text{radius } \mathcal{T}_k^j \cdot \text{vol } \mathcal{T}_k^j \leq 2V_{\max} \coth D_1.$$

Let  $I_2$  be the last term.  $\square$

*Proof of Proposition 5.1.1.* Assume contrarily that  $\text{inj}_{f_k} C_k \rightarrow 0$ , and we will get a contradiction.

Choose the first contact point  $p_k$  on  $\partial \mathcal{T}_k^f$  on which  $f_k$  lies.  $p_k$  may not be the absolute first contact point  $c_k$  since we require that  $p_k$  lies on the component  $\partial \mathcal{T}_k^f$  which might be different from  $\partial \mathcal{T}_k^c$ .  $p_k$  is the point where  $\partial \mathcal{T}_k^f$  either meets the other component of  $\partial \mathcal{T}_k$  or contacts itself. Very locally,  $p_k$  appears as a contact point of two components, one  $\partial \mathcal{T}_k^f$  from the left hand side and the other  $\partial \mathcal{T}_k^{f'}$  from the right hand side. The reference numbers  $f$  and  $f'$  might be the same.

One obvious inequality is

$$(\text{inj } \partial \mathcal{T}_k^f)^2 \leq \text{area } \partial \mathcal{T}_k^f (\leq I_2),$$

and the right hand side of which is bounded by  $I_2$  by the above lemma. Let  $W$  be a constant  $W(D_1, \sqrt{I_2}, -1)$  in Lemma 3.3.2. Then since  $\text{radius } \mathcal{T}_k^f \geq D_1$  and  $\text{inj } \partial \mathcal{T}_k^f \leq \sqrt{I_2}$ , we have

$$\text{inj } \partial \mathcal{T}_k^f \leq W \cdot \text{inj}_{f_k} C_k (\leq W \cdot \text{inj}_{p_k} C_k) \quad (5.1)$$

by Lemma 3.3.2. If we regard  $p_k$  as a point on  $\partial\mathcal{T}_k^{f'}$ , then since radius  $\mathcal{T}_k^{f'} \geq D_1$  and  $\text{inj } \partial\mathcal{T}_k^{f'} \leq \sqrt{I_2}$ , again we have

$$\text{inj } \partial\mathcal{T}_k^{f'} \leq W \cdot \text{inj}_{p_k} C_k. \quad (5.2)$$

Hence  $B_{W \cdot \text{inj}_{p_k} C_k}(C_k, p_k)$  contains a homotopically nontrivial loop  $\ell_1$  based at  $p_k$  on the left  $\partial\mathcal{T}_k^f$ , and also  $\ell_2$  based at  $p_k$  on the right  $\partial\mathcal{T}_k^{f'}$ .

Let  $V(D_1, 2W)$  be a local Margulis constant with respect to  $D_1$  and  $2W$  in Lemma 3.2.1. Since we assumed that  $\text{inj}_{f_k} C_k \rightarrow 0$ , (5.1) and (5.2) imply that

$$(\text{inj}_{p_k} C_k \leq) \text{inj } \partial\mathcal{T}_k^f \leq V(D_1, 2W).$$

for sufficiently large  $k$ . Then  $(B_{2W \cdot \text{inj}_{p_k} C_k}(C_k, p_k), p_k)$  is homeomorphic by an almost equi-expansive map to  $(B_{2W}(E, e), e)$  for some noncompact euclidean manifold  $E$  with  $\text{inj}_e E = 1$  by Lemma 3.2.1 and the remark after that. Furthermore by Lemma 3.2.2, the homomorphism,

$$\pi_1(B_W(E, e), e) \rightarrow \pi_1(B_{2W}(E, e), e),$$

induced by the inclusion have a virtually abelian image. Hence so does

$$\pi_1(B_{W \cdot \text{inj}_{p_k} C_k}(C_k, p_k), p_k) \rightarrow \pi_1(B_{2W \cdot \text{inj}_{p_k} C_k}(C_k, p_k), p_k),$$

because of the choice of a homeomorphism of  $(B_{2W \cdot \text{inj}_{p_k} C_k}(C_k, p_k), p_k)$  we made. Thus the nontrivial loops  $\ell_1, \ell_2$  representing elements of  $\pi_1(B_{W \cdot \text{inj}_{p_k} C_k}(C_k, p_k), p_k)$  are virtually commutative, in particular, in  $\Pi$  for sufficiently large  $k$ .

On the other hand, consider the developed image near  $p_k$ .  $p_k$  lifts to a contact point of lifts of  $\partial\mathcal{T}_k^f$  from the left hand side and that of  $\partial\mathcal{T}_k^{f'}$  from the right hand side. In particular,  $\rho_k(\ell_1)$  leaves the left hand lift of  $\partial\mathcal{T}_k^f$  invariant, on the other hand,  $\rho_k(\ell_2)$  leaves the right hand lift of  $\partial\mathcal{T}_k^{f'}$  invariant. Hence their action on  $\mathbf{H}^3 \cup \mathbf{S}_\infty^2$  do not have common fixed point at all, and they are not commutative even virtually in  $\rho_k(\Pi)$ . This is a contradiction.  $\square$

**5.3. Geometric Limits.** We assume the conclusion of Proposition 5.1.1 that the injectivity radius of the points on  $\partial\mathcal{T}_k$  is uniformly bounded from below by  $I_1 > 0$ , and prove Proposition 5.1.2.

**Lemma 5.3.1.** *Under the conclusion of Proposition 5.1.1, if there is a constant  $D^j > 0$  such that radius  $\mathcal{T}_k^j \leq D^j$ , then  $\beta^j > 0$  and there is a constant  $S^j > 0$  such that*

$$S^j \leq \text{length } \Sigma_k^j.$$

for any  $k$ .

*Proof.* If  $\beta^j = 0$ , then the  $j$ th component  $\alpha_k^j$  of the angle set approaches zero as  $k \rightarrow \infty$ . Thus since radius  $\mathcal{T}_k^j \leq D^j$ ,  $\text{inj } \partial\mathcal{T}_k^j \rightarrow 0$  by (3.5). This contradicts the conclusion of Proposition 5.1.1 since

$$(I_1 \leq) \text{inj}_{f_k} C_k \leq \text{inj } \partial\mathcal{T}_k^j.$$

If  $\{\text{length } \Sigma_k^j\}$  contains a subsequence converging to 0, then since

$$\begin{aligned} \text{area } \partial\mathcal{T}_k^j &= \theta_k^j \text{ length } \Sigma_k^j \sinh \text{radius } \mathcal{T}_k^j \cosh \text{radius } \mathcal{T}_k^j \\ &\leq \pi \sinh D^j \cosh D^j \text{length } \Sigma_k^j, \end{aligned}$$

by (3.2),  $(I_1^2 \leq (\text{inj } \partial \mathcal{T}_k^j)^2 \leq) \text{ area } \partial \mathcal{T}_k^j$  can be arbitrary close to 0. This is again a contradiction.  $\square$

**Lemma 5.3.2.**  *$C_*$  is a hyperbolic 3-cone-manifold of finite volume possibly with compact singularity.*

*Proof.* It is sufficient to show that each point  $x_* \in C_*$  admits a hyperbolic ball neighborhood possibly with singularity along a geodesic segment. The argument is quite parallel to that in Lemma 3.2.1.

Since  $I_1 \leq \text{inj}_{f_k} C_k \leq \text{inj}_{c_k} C_k$ , the limit  $c_* \in C_*$  of the reference points  $\{c_k\}$  admits a neighborhood which is a limit of hyperbolic balls of uniformly bounded radii by  $I_1$ . Hence it admits a hyperbolic ball neighborhood.

Fix a constant  $R > 0$  and choose any  $x_* \in C_*$  with  $d(x_*, c_*) \leq R$ . If it is a limit of points  $\{x_k \in C_k - \mathcal{T}_k\}$ , then since  $d(x_k, \Sigma) \geq D_1$  and we may assume  $d(x_k, c_k) \leq 2R$ ,  $x_k$  admits a hyperbolic ball neighborhood of radius  $\geq U(D_1, I_1, 2R, -1)$  by Lemma 3.1.1, where the radius bound does not depend on  $k$ . Hence again  $x_*$  admits a hyperbolic open ball neighborhood.

If  $x_*$  is a limit of points  $\{x_k \in \mathcal{T}_k\}$ , we may assume that  $\{x_k\}$  is contained in a component  $\mathcal{T}_k^x$  with a constant reference number  $x$  by taking further subsequence if necessary. When radius  $\mathcal{T}_k^x \rightarrow \infty$ , since  $\partial \mathcal{T}_k^x$  does not degenerate by the conclusion of Proposition 5.1.1, we may assume that  $d(x_k, c_k) \leq 2R, d(x_k, \Sigma_k^x) \rightarrow \infty$ , and  $d(x_k, \Sigma_k) \geq D_1$ . Then  $x_k$  admits a hyperbolic ball neighborhood of radius  $\geq U(D_1, I_1, 2R, -1)$  by Lemma 3.1.1 where the radius bound does not depend on  $k$ , and hence so does  $x_*$ . When radius  $\mathcal{T}_k^x$  is bounded not only from below by the Assumption 1 but also from the above, length  $\Sigma_k^x$  has a uniform lower bound away from zero by Lemma 5.3.1, and each point within  $\mathcal{T}_k^x$  has a possibly singular ball neighborhood of uniform radius where the singularity occurs only along a geodesic segment. Hence  $x_*$  admits a hyperbolic ball neighborhood possibly with cone singularity along a geodesic segment.

Since  $R > 0$  was arbitrary, the above argument shows that every point on  $C_*$  admits a hyperbolic ball neighborhood possibly with a cone singularity. The singularity appears only in the limit of  $\mathcal{T}_k^x$  whose radius is bounded. There are only finitely many such components. Moreover the length of a core of such a component is bounded since we have in general by (3.5)

$$(I_1 \leq) \text{inj } \partial \mathcal{T}_k^x \leq \theta_k^x \sinh \text{radius } \mathcal{T}_k^x,$$

so that the formula in (3.2) implies the estimate,

$$\text{length } \Sigma_k^x = \frac{\text{area } \partial \mathcal{T}_k^x}{\theta_k^x \sinh \text{radius } \mathcal{T}_k^x \cosh \text{radius } \mathcal{T}_k^x} \leq \frac{I_2}{I_1 \cosh \text{radius } D_1}.$$

Hence the singular set is compact.  $\square$

By Corollary 1.2.3,  $C_*$  has finitely many toral ends. Choose disjoint horotoral neighborhoods of the ends of  $C_*$  so that the minimum  $I_3$  of the injectivity radii of  $C_*$  at points on their boundaries is  $\leq I_1$ . We let  $C_*^{\text{cut}}$  be a compact hyperbolic cone-manifold with toral boundary obtained from  $C_*$  by truncating such cusp neighborhoods. We thus have

$$I_3 = \min\{\text{inj}_x C_* | x \in \partial C_*^{\text{cut}}\} \leq I_1.$$

**Lemma 5.3.3.** *There is an approximate homeomorphism  $\varphi_k : C_*^{cut} \rightarrow C_k$  for sufficiently large  $k$ .*

*Proof.* We just repeat the last paragraph in the proof of Lemma 3.2.1. Choose a fine triangulation of  $C_*^{cut}$  by 3-simplices whose faces either are totally geodesic or lie on  $\partial C_*^{cut}$  so that  $\Sigma_*^{cut}$  is contained in the 1-skeleton. Then since  $C_*$  is a geometric limit, we may choose a map from the 0-skeleton of  $C_*^{cut}$  to  $C_k$  for sufficiently large  $k$ , where 4 vertices spanning an oriented simplex in  $C_*^{cut}$  are mapped to 4-vertices spanning a simplex with the same orientation also in  $C_k$  and vertices on  $\Sigma_*^{cut}$  are mapped to points on  $\Sigma_k$ . Then its obvious piecewise linear extension is necessarily an into approximate homeomorphism  $\varphi_k : C_*^{cut} \rightarrow C_k$  for sufficiently large  $k$ .  $\square$

**Lemma 5.3.4.**  *$\varphi_k$  can be modified by an isotopy to a homeomorphism, which we again denote by  $\varphi_k$ , so that each component of  $\varphi_k(\partial C_*^{cut})$  bounds an equidistant tubular neighborhood of either a short geodesic or a component of  $\Sigma_k$  in  $C_k$  for further sufficiently large  $k$ . Moreover the isotopy can be chosen so that the injectivity radii of the components of  $\varphi_k(\partial C_*^{cut})$  is uniformly bounded from below by some positive constant.*

*Proof.* Choose a component  $\partial_0 C_*^{cut}$  of  $\partial C_*^{cut}$  and let  $H_k$  be the image of  $\partial_0 C_*^{cut}$  by  $\varphi_k$ , namely  $H_k = \varphi_k(\partial_0 C_*^{cut})$ . Since the nonsingular part  $N_k = C_k - \Sigma_k$  is irreducible and atoroidal as a 3-manifold by Theorem 1.2.1,  $H_k$  either is incompressible and boundary parallel or bounds a solid torus  $Z_k$  in  $N_k$ . In particular  $H_k$  separates  $N_k$ .

If  $H_k$  is incompressible, it is isotopic to a component of  $\partial \mathcal{T}_k$ . Hence it is isotopic to a horotorus bounding an equidistant tubular neighborhood of a component of  $\Sigma_k$ . Choose a horotorus  $\mathcal{H}_k$  isotopic to a corresponding component of  $\partial \mathcal{T}_k$  so that

$$\text{inj } \mathcal{H}_k = I_3/2.$$

It exists certainly in  $\mathcal{T}_k$  since the minimum of the injectivity radius of components of  $\partial \mathcal{T}_k$  is  $\geq \inf_{f_k} C_k \geq I_1 \geq I_3$ . Moreover, since  $\varphi_k$  does not change injectivity radius very much, it is contained outside the image of  $C_*^{cut}$  by  $\varphi_k$ . Hence we can choose isotopy of  $H_k$  to  $\mathcal{H}_k$  by pushing  $H_k$  outside  $\varphi_k(C_*^{cut})$ . In particular, the isotopy is covered by an isotopy of  $C_*^{cut}$  fixing the complement of a small collar neighborhood of  $\partial_0 C_*^{cut}$ , and the covering isotopy does not affect the other component of  $\partial C_*^{cut}$ .

Suppose next that  $H_k$  bounds a solid torus  $Z_k$  in  $N_k$ . We will show that the solid torus  $Z_k$  bounded by  $H_k$  contains a simple closed geodesic isotopic to the core of  $Z_k$ . To see this, we will first extend  $\varphi_k$  to an embedding  $\hat{\varphi}_k$  of the union of  $C_*^{cut}$  and a collar  $F$  of  $\partial_0 C_*^{cut}$  which lies in the complement of  $C_*^{cut}$ . The choice of  $F$  is rather technical and will be made below. Set

$$I_4^k = \inf\{\text{inj}_x C_k \mid x \in H_k\}.$$

This constant depends on  $k$  in fact, however since injectivity radius does not change very much by an approximation  $\varphi_k$ , we may assume that  $I_4^k$  is bounded from below by some positive constant for sufficiently large  $k$ . Then we set a positive constant  $I_4$  by

$$I_4 = \min\{\inf\{I_4^k\}, I_3\}.$$

We now choose a collar  $F$  of  $\partial_0 C_*^{cut}$  so that the second shortest geodesic on the boundary component  $\partial_1 C_*^{cut}$  of  $F$  other than  $\partial_0 C_*^{cut}$  has length  $\leq I_4/2$ . The same

argument of the previous lemma shows that the extension  $\hat{\varphi}_k$  over  $C_*^{cut} \cup_{\partial_0 C_*^{cut}} F$  exists by taking further sufficiently large  $k$ .

Let  $\ell_1$  and  $\ell_2$  be the shortest two geodesics on  $\partial_1 C_*^{cut}$ . They are not homotopic each other because  $\partial_1 C_*^{cut}$  has an euclidean structure. Since  $\hat{\varphi}_k$  does not change the length very much, we may assume that the length of  $\hat{\varphi}_k(\ell_1)$  and  $\hat{\varphi}_k(\ell_2)$  are  $< I_4$ . The new boundary component  $\hat{H}_k$  of  $\hat{\varphi}_k(F)$  other than  $H_k$  is contained in  $Z_k$ . Since  $\hat{\varphi}_k(\ell_1)$  and  $\hat{\varphi}_k(\ell_2)$  are nonhomotopic loops on  $\hat{H}_k$ , at least one of them, say  $\hat{\varphi}_k(\ell_1)$ , is homotopically nontrivial in  $Z_k$ . Notice that  $\hat{\varphi}_k(\ell_1)$  has length  $< I_4$  and hence the length shortening homotopy of  $\hat{\varphi}_k(\ell_1)$  in Lemma 1.2.4 does not go through the point with injectivity radius  $\geq I_4/2$ . On the other hand, any point on  $H_k$  has injectivity radius  $\geq I_4^k \geq I_4$ . Thus the length shortening homotopy of  $\hat{\varphi}_k(\ell_1)$  stays in  $Z_k$  and  $\hat{\varphi}_k(\ell_1)$  shrinks to some nonzero multiple of a closed geodesic  $\ell$  in  $Z_k$ . Then by Theorem 1.2.1,  $C_k - (\Sigma_k \cup \ell)$  is atoroidal, and in particular  $H_k$  is parallel to a torus bounding an equidistant tubular neighborhood of  $\ell$ .

We would like to isotope  $H_k$  to an equidistant torus to  $\ell$  with uniform intrinsic injectivity radius. To see this, consider an increasing family of equidistant tubular neighborhoods of  $\ell$  in  $Z_k$ . At some first critical radius, the boundary of a neighborhood hits either  $H_k$  or itself. If the later is the case, choose a homotopically nontrivial loop  $g$  in the critical equidistant tubular neighborhood which passes the contact point  $p$ .  $g$  is homotopic to some nonzero multiple of  $\ell$ , say  $\ell^d$ , since it lies in the solid torus  $Z_k$  and  $p$  is a contact point of the critical equidistant tubular neighborhood. Then consider the developed image of the critical equidistant tubular neighborhood. The preimage of  $p$  contains points which cannot be joined by the action of  $\rho_k(\ell)$  since  $p$  is a contact point. Simultaneously, it must be an orbit of a point by the action of  $\rho_k(\ell^d)$  since  $g$  is homotopic to  $\ell^d$ . This is impossible.

Hence the equidistant tubular neighborhood of  $\ell$  grows up to the one  $\mathcal{H}_k$  which touches  $H_k$ . Since each point of  $H_k$  has injectivity radius  $\geq I_4$  in  $C_4$ , we have

$$\text{inj } \mathcal{H}_k > I_4.$$

Moreover, it is contained outside the image of  $C_*^{cut}$  by  $\varphi_k$ . Hence we can choose an isotopy of  $H_k$  to  $\mathcal{H}_k$  by pushing  $H_k$  into  $Z_k$ . In particular, the isotopy is covered by an isotopy of  $C_*^{cut}$  fixing the complement of a small collar neighborhood of  $\partial_0 C_*^{cut}$ . The covering isotopy does not affect the other component of  $\partial C_*^{cut}$  and we are done.  $\square$

**Lemma 5.3.5.** *For any geodesic loop  $\ell$  on  $\partial C_*^{cut}$ ,  $\text{length } \varphi_k(\ell)$  is bounded by some positive constants from both above and below for all  $k$ , where  $\varphi_k$  is a modified one in the previous lemma.*

*Proof.* The existence of a lower bound is a simple corollary to the previous lemma. Recall that the isotopy we constructed pushes the boundary into an equidistant tubular neighborhood. Thus it is distance decreasing on the boundary. Hence since the original  $\varphi_k|_{\partial_0 C_*^{cut}} : \partial_0 C_*^{cut} \rightarrow H_k$  is an approximation and does not change the distance very much even when  $k$  varies, a uniform upper bound exists.  $\square$

By taking a further subsequence and rearranging reference numbers of the components of  $\Sigma$  if necessary, we may assume that the  $j$ th component of  $\partial C_*^{cut}$  is mapped by  $\varphi_k$  to a torus bounding an equidistant tubular neighborhood of  $\Sigma_k^j$  for  $0 \leq j \leq s$  and of a short geodesic in  $C_k$  for  $s < j \leq t$  for sufficiently large  $k$ .



The following lemma finishes the proof of Proposition 5.1.2.

**Lemma 5.3.6.** *There are no components of  $\partial C_*^{cut}$  which are mapped by  $\varphi_k$  to the torus bounding an equidistant tubular neighborhood of a short geodesic. In other words,  $s = t$ , and there are no cusp openings away from the singularity.*

*Proof.* Assume contrarily that  $s < t$ . Filling the  $j$ th component of the boundary of  $C_*^{cut}$  for each  $s < j \leq t$  by an equidistant tubular neighborhood of a short geodesic which the  $j$ th component of  $\varphi_k(\partial C_*^{cut})$  bounds in  $C_k$ , we obtain a cone-manifold homeomorphic to  $C - \cup_{j \leq s} \Sigma^j$ . In other words, for each sufficiently large  $k$ ,  $C_*^{cut}$  produces an isometric hyperbolic cone-manifold by some Dehn filling on the last  $t - s$  components of  $\partial C_*^{cut}$ .

If the number of slopes appeared in this Dehn filling on the  $j$ th component is finite even as  $k$  varies, then by taking a further subsequence, we may assume that the slope is unique and does not depend on  $k$ . Denote the geodesic representative of a slope on  $\partial C_*^{cut}$  by  $m$ . Then by Lemma 5.3.5,  $\text{length } \varphi_k(m)$  is bounded from above by some constant not depending on  $k$ . On the other hand, since the equidistant tubular neighborhood of a short geodesic bounded by the  $j$ th component of  $\varphi_k(\partial C_*^{cut})$  approaches a cusp, and since the cone angles of short geodesics stay  $2\pi$ , the lengths of curves on  $\varphi_k(\partial C_*^{cut})$  bounding a disk in tubes must diverge by Lemma 3.3.1. This is a contradiction.

Hence the number of slopes appeared in the Dehn fillings must be infinite for each component when  $k$  varies. Let us reconsider the situation not by cutting toral ends but by removing the singularity. Come back to the limiting cone-manifold  $C_*$ . The nonsingular part  $N_* = C_* - \Sigma_*$  admits a hyperbolic structure by Theorem 1.2.1. We denoted it by  $C_{*,comp}$ . Then for each  $k$ ,  $C_{*,comp}$  produces  $C_{comp}$  by some Dehn filling on the last  $t - s$  cuspidal components of  $C_{*,comp}$ . The filling slope on each component varies infinitely many as  $k$  varies. Hence such filling slopes accumulate to  $\infty$  which corresponds to the complete structure  $C_{*,comp}$ . This means that  $C_{*,comp}$  produces  $C_{comp}$  by infinitely many hyperbolic Dehn fillings. On the other hand, the slopes of hyperbolic Dehn fillings on a fixed hyperbolic manifold which produce the same manifold is only finitely many. This can be verified for instance by listing volumes (see [14]). Hence we get a contradiction.  $\square$

**5.4. Algebraic Limits.** We assume the conclusion of Proposition 5.1.1 throughout this subsection and prove Proposition 5.1.3 by lemmas below. We continue to use  $C_*^{cut}$  in the previous subsection.  $C_*^{cut}$  has  $s$  toral boundaries and each  $\varphi_k : C_*^{cut} \rightarrow C_k$  maps the  $j$ th component of  $\partial C_*^{cut}$  to a torus bounding an equidistant tubular neighborhood of  $\Sigma_k^j$ . The cusp opening on  $C_*$  does not simply imply  $(\beta^j =) \lim_{k \rightarrow \infty} \theta_k^j = 0$  for  $j \leq s$  as we pointed out in section 3.3. However in this case, we have

**Lemma 5.4.1.**  $\beta^j = 0$  for all  $1 \leq j \leq s$ .

*Proof.* Suppose contrarily that  $\beta^j > 0$  for some  $1 \leq j \leq s$  and choose a meridional element  $m_k$  bounding  $\Sigma_k^j$  on  $\varphi_k(\partial C_*^{cut})$  for each  $k$ . If  $\{\varphi_k^{-1}(m_k) \mid k\}$  contains only finitely many isotopy classes of the curves on  $\partial C_*^{cut}$ , then  $\{\text{length } \varphi_k^{-1}(m_k) \mid k\}$  is bounded by Lemma 5.3.5. On the other hand, since a singular solid torus bounded by this component of  $\varphi_k(\partial C_*^{cut})$  approaches a cusp, the lengths of meridional elements on the boundary must diverge by Lemma 3.3.1. This is a contradiction.

Thus it is enough to show that  $\{\varphi_k^{-1}(m_k) \mid k\}$  contains only finitely many isotopy classes of the curves on  $\partial C_*^{cut}$ . Composing reference homeomorphisms of (1) in 4.3, we obtain a homeomorphism  $\xi_k \circ \xi_{k_0}^{-1} : C_{k_0} \rightarrow C_k$ . Then  $\xi_k \circ \xi_{k_0}^{-1}(m_{k_0})$  is isotopic to  $m_k$ . Fixing  $k_0$  and running  $k > k_0$ , we have infinitely many homeomorphisms  $\psi_k = \varphi_k^{-1} \circ \xi_k \circ \xi_{k_0}^{-1} \circ \varphi_{k_0} : C_*^{cut} \rightarrow C_*^{cut}$ . Since the interior of  $C_*^{cut}$  is homeomorphic to  $C_*$ ,  $C_*^{cut}$  admits only finitely many isotopy classes of automorphisms by Corollary 1.2.2. Hence there are only finitely many isotopy classes in  $\{\psi_k \mid k > k_0\}$ . This is enough since  $\psi_k(\varphi_{k_0}^{-1}(m_{k_0}))$  is isotopic to  $\varphi_k^{-1}(m_k)$ .  $\square$

**Lemma 5.4.2.** *The angle set of  $C_*$  is equal to  $B$ .*

*Proof.* Lemma 5.4.1 shows that  $\beta^j = 0$  for  $j \leq s$ . Hence it is equal to the  $j$ th component of the angle set of  $C_*$  since the component corresponds to a cusp.

The other component corresponds to a component of  $\Sigma_*$  in  $C_*$ . As we have seen in the proof of Lemma 5.3.2, a tubular neighborhood of  $\Sigma_*^j$  for  $j > s$  is a limit of  $\mathcal{T}_k^j$ 's whose cone angles are  $\alpha_k^j$ 's. Thus the cone angle of  $\Sigma_*^j$  in this case is  $\lim_{k \rightarrow \infty} \alpha_k^j$  which equals  $\beta^j$  by definition.  $\square$

**Lemma 5.4.3.** *A sequence  $\{\rho_k\}$  of holonomy representations of  $\{C_k\}$  contains a subsequence converging algebraically to the holonomy representation of  $C_*$  with respect to the identification by  $\xi_k$ .*

*Proof.* We eventually obtained an into homeomorphism  $\varphi_k : (C_*^{cut}, \Sigma_*^{cut}) \rightarrow (C_k - \cup_{j \leq s} \Sigma_k^j, \cup_{j > s} \Sigma_k^j)$  for sufficiently large  $k$  by the lemmas in the previous subsection. On the other hand, there are reference homeomorphisms  $\xi_k : C \rightarrow C_k$  of (1) in 4.3. Then the composition  $\xi_k^{-1} \circ \varphi_k : (C_*^{cut}, \Sigma_*^{cut}) \rightarrow (C - \cup_{j \leq s} \Sigma^j, \cup_{j > s} \Sigma^j)$  is an into homeomorphism. There are only finitely many isotopy classes of such maps since otherwise,  $C$  would admit an infinite automorphism group, contradicting Corollary 1.2.2. Hence taking a further subsequence, we may assume that  $\xi_k^{-1} \circ \varphi_k$ 's are isotopic for all  $k$ . Then the algebraic convergence is a consequence of a geometric convergence.  $\square$

## 6. THIN TUBE

In this section, we study what happens to a geometric limit  $C_*$  of a sequence of deformations  $\{C_k\}$  in 4.3 when the minimum of maximal tube radii goes to zero. The analysis involves noncompact euclidean 3-cone-manifolds with noncompact singularity, whose definition would be obvious.

**6.1. Brief Outline.** The minimum of radius  $\mathcal{T}_k^j$  is attained by the  $c$ th component  $\mathcal{T}_k^c$  which contains the first contact point  $c_k$ . The underlying assumption throughout this section is

**Assumption 2.** *If  $k \rightarrow \infty$ , then*

$$\text{radius } \mathcal{T}_k^c \rightarrow 0$$

Under the Assumption 2 above, we discuss possible degenerations in two propositions below. The technical assumption there will be satisfied by some natural setting which we will use later on.

**Proposition 6.1.1.** *Under the Assumption 2, if there is a constant  $V_{\min} > 0$  such that  $\text{vol } C_k \geq V_{\min}$ , and if  $\beta^j$  is strictly less than  $\pi$  for all  $1 \leq j \leq n$ , then  $C_*$  is isometric to either the euclidean line  $\mathbf{E}$  or the half line  $\mathbf{E}_{\geq 0}$ .*

**Remark .** We do not know whether  $\mathbf{E}_{\geq 0}$  really occurs as a geometric limit.

**Proposition 6.1.2.** *If furthermore  $\beta^c > 0$ , then the rescaling limit  $\overline{C}_*$  of  $\{C_k\}$  by normalizing the radius of  $\mathcal{T}_k^c$  to be 1 is an euclidean cone-manifold isometric to  $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$ , where  $\mathbf{S}^2(\alpha, \beta, \gamma)$  is an euclidean 2-cone-manifold over the 2-sphere with three cone points of cone angles  $\alpha, \beta, \gamma$  such that  $0 < \alpha, \beta, \gamma < \pi$  and  $\alpha + \beta + \gamma = 2\pi$ .*

**Remark .** It is quite unlikely that both radius  $\mathcal{T}_k^c \rightarrow 0$  and  $\beta^c = 0$  occur simultaneously, though we do not have a proof.

**6.2. Collapsing.** Consider the Dirichlet polyhedron  $P_{c_k}$  of  $C_k$  about the first contact point  $c_k$ , which we simply denote by  $P_k$  from now on. Supposing that the Assumption 2 is the case, we analyze the limit of  $P_k$  and prove Proposition 6.1.1.

**Lemma 6.2.1.** *Under the assumption of Proposition 6.1.1,  $\{(P_k, c_k)\}$  converges geometrically to the euclidean line  $\mathbf{E}$  or the half line  $\mathbf{E}_{\geq 0}$ .*

*Proof.* Imagine that  $c_k$  is the contact point of  $\mathcal{T}_k^c$  from the top side and  $\mathcal{T}_k^{c'}$  from the bottom sides. The reference number  $c$  might be equal to  $c'$ . The shortest common orthogonal to  $\Sigma_k^c$  and  $\Sigma_k^{c'}$  which goes through  $c_k$  lifts to the geodesic segment  $g_k \subset P_k$ . It is in fact the segment realizing the length = 2 radius  $\mathcal{T}_k^c$ , and by the Assumption 2, length  $g_k$  goes to zero as  $k \rightarrow \infty$ .

Let  $p_k, q_k$  be terminal points of  $g_k$ . Since we assumed that  $\beta^j$  is strictly less than  $\pi$ ,  $P_k$  is locally bounded by roof shaped faces near  $p_k$  from the top and  $q_k$  from the bottom respectively, where their ridges correspond to  $\Sigma_k^c$  and  $\Sigma_k^{c'}$ .  $P_k$  is convex, and it is bounded by the extension of these roofs from the top and bottom. Moreover since length  $g_k \rightarrow 0$ , and since the volume is assumed to be bounded away from zero, the roof ridges become arbitrary close and parallel. Hence  $\{P_k\}$  converges as a metric space to a connected closed subset of the euclidean line  $\mathbf{E}$ .  $\square$

*Proof of Proposition 6.1.1.* Choose for each  $k$  a segment  $l_k \subset \mathbf{E}$  through  $c_k$  so that it is maximally embedded in  $P_k$ . By the previous lemma, we have length  $l_k \rightarrow \infty$  when  $k \rightarrow \infty$ . Thus a long segment  $l_k$  can be isometrically embedded in  $C_k$ . Assigning to each point of  $C_k$  the nearest point on the image of  $l_k$ , we obtain a map  $\varphi_k : C_k \rightarrow l_k \subset \mathbf{E}$ . Then the relation  $R_k$  between  $C_k$  and  $\mathbf{E}$  defined by  $R_k = \{(x, y) \in C_k \times \mathbf{E} | \varphi_k(x) = y\}$  is an approximation for some  $\varepsilon$  where  $\varepsilon \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

**6.3. Rescaling.** In this subsection, under the conclusion of Proposition 6.1.1, we prove Proposition 6.1.2.

**Lemma 6.3.1.** *If  $\beta^c > 0$ , then the rescaling limit of  $\{C_k\}$  normalizing the radius of  $\mathcal{T}_k^c$  to be 1 is a noncompact euclidean cone-manifold with nonempty singular set.*

*Proof.* Multiply  $1/\text{radius } \mathcal{T}_k^c$  on the metric of  $C_k$ , we obtain a cone-manifold  $\overline{C}_k^c$  of constant curvature =  $-(\text{radius } \mathcal{T}_k^c)^2$ , which is  $\geq -1$  for large  $k$ . Then radius  $\overline{\mathcal{T}}_k^c =$

1. Notice that the estimate (3.5) is in fact valid for tubes with constant sectional curvature  $= K$  where  $-1 \leq K \leq 0$ , because the bound in the hyperbolic case is the worst. Since  $\beta^c$  is strictly less than  $\pi$ ,  $\text{inj } \partial \overline{\mathcal{T}}_k^c \leq \pi \sinh 1$  by this new estimate. Then by Lemma 3.3.2, we have a constant  $W = W(1, \pi \sinh 1, -1)$  such that

$$\text{inj } \partial \overline{\mathcal{T}}_k^c \leq W \cdot \text{inj}_{\overline{\mathcal{C}}_k} \overline{\mathcal{C}}_k.$$

Since the  $c$ th component  $\beta^c$  of the angle set is positive by the assumption, and length  $\overline{\Sigma}_k^c$  diverges, we can embed an euclidean disk of radius  $\beta^c/2$  into  $\partial \overline{\mathcal{T}}_k^c$  by the euclidean case of (3.1). Therefore,  $\text{inj } \partial \overline{\mathcal{T}}_k^c \geq \beta^c/2$  for sufficiently large  $k$ ,  $\text{inj}_{\overline{\mathcal{C}}_k} \overline{\mathcal{C}}_k$  is uniformly bounded from below, and in particular,  $\overline{\mathcal{C}}_*$  admits an euclidean ball neighborhood.

On the other hand, each nonsingular component  $\Sigma_k^j$  either becomes parallel to  $\Sigma_k^c$  in  $C_k$  or goes far away from  $c_k$ . In particular, either length  $\Sigma_k^j \rightarrow \infty$  or  $d(c_k, \Sigma_k^j) \rightarrow \infty$ . This is true also in the rescaled setting.

These two informations are good enough to conclude that  $\{(\overline{\mathcal{C}}_k, \overline{\mathcal{C}}_k)\}$  converges geometrically to an euclidean cone-manifold  $(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  since the singularity admits uniformly thick tubular neighborhood and its length does not degenerate, also the reference point stays in a uniformly thick part. The singular set is nonempty because  $\overline{\Sigma}_*$  has distance 1 to  $\overline{\mathcal{C}}_*$ .  $\square$

**Lemma 6.3.2.**  $\overline{\mathcal{C}}_*$  has two ends.

*Proof.* That  $\overline{\mathcal{C}}_*$  has two ends is equivalent to that  $C_*$  converges to  $\mathbf{E}$  instead of  $\mathbf{E}_{\geq 0}$ .

Assume contrarily that  $C_*$  converges to  $\mathbf{E}_{\geq 0}$  and choose  $R > 0$  sufficiently large so that  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  is connected.  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  can be seen in the Dirichlet polyhedron  $\overline{P}_*$  of  $\overline{\mathcal{C}}_*$  about  $\overline{\mathcal{C}}_*$  as an intersection of  $\overline{P}_*$  and the sphere of radius  $= R$ . The faces of  $\overline{P}_*$  intersecting  $\partial B_R(\mathbf{E}^3, \overline{\mathcal{C}}_*)$  for large  $R$  all must be parallel to the ray to the end, and hence the combinatorial structure of  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  is very simple. It consists of one 2-cell  $e$  with  $\mu$  edges, where  $\mu$  is equal to the number of faces of  $\overline{P}_*$  intersecting  $\partial B_R(\mathbf{E}^3, \overline{\mathcal{C}}_*)$ . Also the topology of  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  does not change for sufficiently large  $R$  since  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  tends to be orthogonal to the ray to the end.

To see more about vertices, we let  $\nu_i$  be the total angle of corners of  $e$  surrounding the  $i$ th vertex of  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  and  $\nu$  the number of vertices. Then by Gauss-Bonnet, we have the identity.

$$\begin{aligned} - \int_{\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)} K_R dA &= \int_{\partial e} \kappa_g ds + (\mu - 2)\pi - \sum_{i=1}^{\nu} \nu_i \\ &= \int_{\partial e} \kappa_g ds - 2\pi\chi(\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)) + \sum_{i=1}^{\nu} (2\pi - \nu_i), \end{aligned}$$

where  $K_R$  is a Gaussian curvature of  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  supported on the interior of  $e$  and  $\kappa_g$  is a geodesic curvature along  $\partial e$ .

Let us see what happens when  $R \rightarrow \infty$ . The left hand side goes to zero since  $K_R \rightarrow 0$  and area  $\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*)$  is bounded. The first term of the right hand side approaches zero also since  $\kappa_g \rightarrow 0$  but the length of  $\partial e$  is bounded. For each vertex not on  $\Sigma_*$ ,  $\nu_i \rightarrow 2\pi$ , and on  $\Sigma_*$ ,  $\nu_i \rightarrow \beta^j$  where  $\beta^j$  is a cone angle of the singularity on which the  $i$ th vertex lies. Hence if  $R$  is large enough, the contribution of  $2\pi\chi(\partial B_R(\overline{\mathcal{C}}_*, \overline{\mathcal{C}}_*))$ ,

which is constant, and the contribution of the cone angles of  $\Sigma_* \cap \partial B_R(\overline{C}_*, \overline{c}_*)$  are identical.

Since we have assumed that  $0 \leq \beta^j < \pi$  for all  $j$  but  $c$  and  $0 < \beta^c < \pi$ , this cancellation occurs only when  $\chi(\partial B_R(\overline{C}_*, \overline{c}_*)) = 2$  and  $\Sigma_* \cap \partial B_R(\overline{C}_*, \overline{c}_*)$  consists of three points. This is a contradiction since a noncompact euclidean cone-manifold with one end must have even number of ends of singularity.  $\square$

The following classification of noncompact euclidean 3-cone-manifolds with two ends whose cone angles all are  $\leq \pi$  finishes the proof of Proposition 6.1.2.

**Lemma 6.3.3.** *An orientable euclidean 3-cone-manifold  $E$  with nonempty singular set of cone angles  $\leq \pi$  and with two ends is a product of a compact euclidean 2-cone-manifold and  $\mathbf{E}$ . More precisely,  $E$  is isometric to either*

1.  $\mathbf{S}^2(\pi, \pi, \pi, \pi) \times \mathbf{E}$  or
2.  $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$ , where  $\alpha + \beta + \gamma = 2\pi$ .

*Proof.* Choose a reference point  $e \in E$ . A Dirichlet polyhedron  $P_e$  is a convex polyhedron. Since  $E$  has two ends, there are two rays  $r_1, r_2$  in  $P_e$  based at  $e$ . If  $r_1 \cup r_2$  had bent at  $e$ , then  $P_e$  cannot have two ends. Hence  $r_1 \cup r_2$  is a straight line. Then any face  $P_e$  must be parallel to  $r_1 \cup r_2$  by convexity and they surround  $r_1 \cup r_2$ . Let  $Q$  be a polygon through  $e$  which intersects perpendicularly to the faces. It must be compact since otherwise,  $P_e$  would not have two ends. Then  $\partial Q$  is glued with  $\partial Q$  via the identification of  $P_e$  because  $P_e$  is starlike and hence the identifications do not contain any translation factor along  $r_1 \cup r_2$ . Thus  $Q$  becomes an euclidean sub-cone-manifold after identification and  $E$  is the product of this sub-cone-manifold and  $\mathbf{E}$ .

The remaining is to classify compact euclidean 2-cone-manifolds with cone angles  $\leq \pi$ . However this is a routine application of Gauss-Bonnet, so that we leave it to the reader.  $\square$

## 7. CONTINUOUS FAMILIES

In this section, we come back to a continuous family of deformations of a compact orientable hyperbolic 3-cone-manifold  $C$ , see what happens in the limit with the aid of the propositions in the previous sections, and prove the main theorem and its corollaries.

**7.1. Brief Outline.** Let  $\{C_\theta\}$  be a continuous family of deformations of  $C$  parameterized by the angle assignment

$$\theta : [0, 1) \rightarrow \Theta = (\mathfrak{Z}\mathbf{C})^n$$

where

$$\lim_{t \rightarrow 1} \theta(t) = B = (\beta^1, \dots, \beta^n).$$

We will first of all generalize the concept of strong convergence of a sequence to a continuous family of deformations.

**Definition .** A continuous family  $\{C_\theta\}$  of the deformations of  $C$  is said to converge strongly to  $C_*$  if every subsequence  $\{C_k\}$  in  $\{C_\theta\}$  whose angle set tends to  $B$  converges strongly to  $C_*$ .

The following theorems, which we will prove in this section, are what we can conclude for angle monotone families from the analysis of the previous sections. As we will see later in the proof, the strong convergence of a family is very much likely derived from a strong convergence of a sequence together with the local rigidity.

**Theorem 7.1.1.** *If the family  $\{C_\theta\}$  has a component-wise decreasing angle assignment  $\theta$  and if  $\beta^j > 0$  for all  $1 \leq j \leq n$ , then  $\{C_\theta\}$  converges strongly to a compact hyperbolic cone-manifold  $C_*$  homeomorphic to  $C$ .*

**Remark .** The positivity of  $\beta^j$  is conjecturally unnecessary. However, the proof we present depends on Proposition 6.1.2 which involves this unclear hypothesis. Also it forces us to make a technical arrangement in the proof of the main theorem.

**Theorem 7.1.2.** *If the family  $\{C_\theta\}$  has a component-wise increasing angle assignment  $\theta$  and  $\beta^j < \pi$  for all  $1 \leq j \leq n$ , then either*

1.  $\text{vol } C_\theta \rightarrow 0$ ,
2.  $C_\theta$  contains a sub-cone-manifold homeomorphic to  $\mathbf{S}^2$  with three cone points so that the sum of their cone angles approaches  $2\pi$  as  $\theta \rightarrow B$ , or
3.  $\{C_\theta\}$  converges strongly to a hyperbolic cone-manifold homeomorphic to  $C$ .

The first theorem will be used to prove the main results in the final subsection. The second theorem above is just to note what we can conclude for angle increasing family, and it is not related to the main results directly.

**7.2. Splitting.** The following example shows the degeneration discussed in Proposition 6.1.2 really occurs in an angle increasing continuous family.

**Example .** Let  $\alpha, \beta, \gamma$  be numbers such that  $0 < \alpha, \beta, \gamma < \pi$  and  $\alpha + \beta + \gamma = 2\pi$ . Consider a hyperbolic tetrahedron whose dihedral angles along ridges are  $(\alpha - \varepsilon)/2$ ,  $(\beta - \varepsilon)/2$ ,  $(\gamma - \varepsilon)/2$  in three opposite pairs, where  $\varepsilon$  is a small nonnegative number. When  $\varepsilon = 0$ , the tetrahedron has four ideal vertices. If  $\varepsilon > 0$ , then the tetrahedron is of infinite volume. It becomes compact by truncating the ends by polar planes. The result is called a truncated tetrahedron. We thus obtain a family of polyhedra  $\{\Delta_\varepsilon(\alpha, \beta, \gamma)\}$  of finite volume parameterized by  $\varepsilon \geq 0$ .

When  $\varepsilon > 0$ , taking a double of  $\Delta_\varepsilon(\alpha, \beta, \gamma)$  along 4 hexagonal faces, we obtain a compact hyperbolic cone-manifold with geodesic boundary. Taking further double along the boundary, we obtain a family of closed hyperbolic 3-cone-manifolds  $\{C_\varepsilon(\alpha, \beta, \gamma)\}$ . The singular set  $\Sigma$  consists of 6 circles each of which is assigned  $\alpha - \varepsilon$ ,  $\beta - \varepsilon$ ,  $\gamma - \varepsilon$  in pairs as their cone angles. If  $\varepsilon \rightarrow 0$ , then the face of truncation becomes very tiny, and  $C_\varepsilon(\alpha, \beta, \gamma)$  splits into two noncompact hyperbolic cone-manifolds by tearing off the boundary of the first double. This family shows that the degeneration in Proposition 6.1.2 certainly occurs at the face of truncation. The reference point lies on the face of truncation, and the rescaling geometric limit is isometric to  $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$ .

This example gives us a fairly general picture. In fact, using the following observation, we will show in the proof of Theorem 7.1.1 that the splitting degeneration caused by appearance of an euclidean sub-cone-manifold such as this cannot occur in angle decreasing families.

**Lemma 7.2.1.** *Let  $\iota_1, \iota_2$  be elliptic elements in  $\mathrm{PSL}_2(\mathbf{C})$  with axis  $\ell_1, \ell_2$  respectively. If  $\iota_1 \iota_2$  is elliptic with axis  $\ell_{12}$ , and if the total angle of rotations of  $\iota_1, \iota_2$  and  $\iota_1 \iota_2$  is  $> 2\pi$ , then  $\ell_1, \ell_2$  and  $\ell_{12}$  meets at the unique point in  $\mathbf{H}^3$ .*

*Proof.* It is not quite hard to show that  $\iota_1 \iota_2$  is loxodromic if the union  $\ell_1 \cup \ell_2$  does not lie on a geodesic plane in  $\mathbf{H}^3$ . Hence we may assume that  $\ell_1 \cup \ell_2$  lies on a geodesic plane  $X$ . Replacing the role of  $\iota_1, \iota_2$  by  $\iota_1^{-1}, \iota_1 \iota_2$  and  $\iota_1 \iota_2, \iota_2^{-1}$ , we get geodesic planes  $Y$  and  $Z$  supporting  $\ell_1 \cup \ell_{12}$  and  $\ell_{12} \cup \ell_2$  respectively. If three planes  $X, Y, Z$  meets in  $\mathbf{H}^3$ , then we are done. If not, they either meet at  $\mathbf{S}_\infty^2$  or does not meet in  $\mathbf{H}^3 \cup \mathbf{S}_\infty^2$  and admits a geodesic plane meeting  $X, Y, Z$  perpendicularly. In both cases, the sum of three rotation angles must be  $\leq 2\pi$  and the assumption is not satisfied.  $\square$

**7.3. Angle Decreasing Family.** In this subsection, we prove Theorem 7.1.1 using propositions in the previous sections and the observation in Lemma 7.2.1. First of all, we have a lower bound of the volume.

**Lemma 7.3.1.** *If  $\theta$  is component-wise decreasing, then there is a constant  $V_{\min} > 0$  such that*

$$V_{\min} \leq \mathrm{vol} C_{\theta(t)}$$

for all  $t \in [0, 1)$ .

*Proof.* Recall Schläfli's variation formula revisited by Hodgson [6],

$$d \mathrm{vol} C_\theta = -\frac{1}{2} \sum_{j=1}^n \mathrm{length} \Sigma^j d\theta^j,$$

where  $\theta^j$  is the  $j$ th component of  $\theta$ . It says that the volume is an increasing function in angle decreasing deformations. Hence  $\mathrm{vol} C_{\theta(t)} \geq \mathrm{vol} C_{\theta(0)} = \mathrm{vol} C$ .  $\square$

*Proof of Theorem 7.1.1.* Given an angle decreasing family  $\{C_\theta\}$ , where  $\beta^j > 0$  for  $1 \leq j \leq n$ , we set  $C_i = C_{\theta(1-1/i)}$  and choose a geometrically convergent subsequence  $\{C_k\}$  in 4.3 with canonical reference homeomorphisms  $\{\xi_k\}$ .

Assume that  $\{C_k\}$  satisfies the Assumption 2. Since the volume is bounded from below, and also since  $0 < \beta^c < \pi$ , the rescaling geometric limit  $\overline{C}_*$  is by Proposition 6.1.2 isometric to the product  $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$  where  $\alpha + \beta + \gamma = 2\pi$ . Since  $\overline{C}_*$  contains an euclidean 2-cone-manifold as a section, we can find a topologically same section by an approximation in a reasonably large neighborhood of  $c_k$ , say  $B_R(C_k, c_k)$ , for sufficiently large  $k$ . It is homeomorphic to the 2-sphere transversely intersecting  $\Sigma_k$  at three points. The total sum of cone angles of these points is more than  $2\pi$  since the deformation is angle decreasing.

Now, there are two components  $\ell_1$  and  $\ell_2$  of  $B_R(C_k, c_k) \cap \Sigma_k$  which admit the shortest common orthogonal  $g$  going through  $c_k$ . Develop  $\ell_1 \cup g \cup \ell_2$ , then the images of  $\ell_1$  and  $\ell_2$  cannot have a common point even in their extensions. On the other hand, if we let two meridional elements rounding  $\ell_1$  and  $\ell_2$  be  $m_1$  and  $m_2$  respectively, then since  $m_1 m_2$  becomes a meridional element rounding the last component,  $\rho_k(m_1 m_2)$  represents an elliptic element. Moreover the total angles of rotations of  $\rho_k(m_1), \rho_k(m_2)$  and  $\rho_k(m_1 m_2)$  is  $> 2\pi$ . Thus by Lemma 7.2.1, the developed image of  $\ell_1$  and  $\ell_2$  must have common point in their extensions. This is a contradiction.

Hence  $\{C_k\}$  does not satisfy the Assumption 2, and the radius of the maximal tube must be uniformly bounded away from zero. We can now apply the results in section 5. In particular, the geometric limit  $C_*$  is a strong limit of a sequence  $\{C_k\}$  by Corollary 5.1.4.

To see that  $C_*$  is a strong limit of a family  $\{C_\theta\}$ , let  $\rho_*$  be a holonomy representation of  $C_*$ . Since it is a holonomy representation of a cone-manifold  $C_*$ , it can be deformed in a small range by Corollary 2.1.2. Let us choose a small path on the space of representations  $\text{Hom}(\Pi, \text{PSL}_2(\mathbf{C}))/\text{PSL}_2(\mathbf{C})$  from  $\rho_*$  supported on  $[0, \varepsilon)$  so that the associated angle assignment is equal to  $\theta(1 - t)$  where  $t \in [0, \varepsilon)$ . This path and the path defined by  $\{\rho_{\theta(t)}\}_{0 \leq t < 1}$  in the space of representations have common points accumulating  $\rho_*$ , which are realized by holonomy representations  $\{\rho_k\}$  of  $\{C_k\}$  in 4.3. Then since the paths are the image of the same angle assignment, they must be the same by the local rigidity at  $\rho_*$ .  $\square$

**7.4. Angle Increasing Family.** In this subsection, we prove Theorem 7.1.2 and present one example for which the theorem can be applied.

*Proof of Theorem 7.1.2.* Assume that  $\text{vol } C_\theta$  does not converge to zero, in other words, (1) is not the case. Choose a sequence  $\{(C_k, c_k)\}$  as in 4.3. If the Assumption 1 is the case, then by Corollary 5.1.4, we have a strong limit  $C_*$  of  $\{C_k\}$ .  $C_*$  is also a strong limit of a family  $\{C_\theta\}$  by the same argument in Theorem 7.1.1, and we get the third case. If the Assumption 2 is the case, then by Proposition 6.1.2, the rescaling limit  $\overline{C}_*$  is isometric to  $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$ . Thus  $\overline{C}_*$  contains an euclidean sub-cone-manifold as its section. We then have a topologically same section in  $C_k$  for sufficiently large  $k$  by an approximation and hence we are in the second case.  $\square$

**Example .** This observation can be used for example to study an angle increasing family  $\{\mathbf{8}_\theta\}$  on the 3-sphere singular along the figure eight knot (see [14]). Since an underlying space of  $\mathbf{8}_\theta$  is the 3-sphere, (2) does not occur. Hence the angle increasing deformation is possible as long as  $\text{vol } \mathbf{8}_\theta > 0$ . The  $A$ -polynomial in [1], which is

$$-M^4 + L(M^8 - M^6 - 2M^4 - M^2 + 1) - L^2M^4 \quad (7.1)$$

for the figure eight knot for instance, represents a relation between an eigenvalue  $M$  of an meridian and an eigenvalue  $L$  of a longitude for  $\text{SL}_2(\mathbf{C})$ -representations of a knot group. Then setting  $M = \exp(t\sqrt{-1}/2)$  in the equation (7.1) = 0, we obtain

$$\cosh \log(-L) = -\frac{L + L^{-1}}{2} = 1 + \cos t - \cos 2t.$$

This shows that  $L$  is always real and the length of the singularity at  $t$  is equal to  $2 \log(-L)$ . Thus by Schläfli's formula,

$$\begin{aligned} \text{vol } \mathbf{8}_\theta &= -\int_0^\theta \log(-L) dt + \text{vol } \mathbf{8}_0 \\ &= -\int_0^\theta \text{arccosh}(1 + \cos t - \cos 2t) dt - 6 \int_0^{\pi/3} \log |2 \sin t| dt \end{aligned}$$

Hence to find the deformable range is reduced to the computation of this integral. A numerical computation shows that  $\mathbf{8}_\theta$  survives as long as  $\theta < 2\pi/3$ .



### 7.5. Proof of Theorem and Corollaries.

*Proof of Theorem.* Given a compact orientable hyperbolic 3-cone-manifold  $(C, \Sigma)$  with an angle set  $A = (\alpha^1, \dots, \alpha^n)$ , we start with a complete structure  $C_{comp}$  supported on the nonsingular part  $N = C - \Sigma$ . Since a small angle changing deformation of  $C_{comp}$  uniquely exists by Corollary 2.1.2, there is an angle set  $B = (\beta^1, \dots, \beta^n)$  very close to  $(0, \dots, 0)$  so that  $C_{comp}$  admits angle increasing deformations along a linear path  $\zeta : [0, 1] \rightarrow \Theta$  with  $\zeta(1) = B$ . Moreover we can choose each  $\beta^j$  positive so that  $\beta^j = 2\pi/b_j$  ( $< \alpha^j$ ) for some large integer  $b_j$ . Let  $\{C_{\zeta(t)}\}$  be the associated family of deformations. The cone-manifold  $C_{\zeta(1)}$  with the angle set  $B$  shares the topological type with  $C_{comp}$  and hence  $C$ . Moreover  $C_{\zeta(1)}$  is an orbifold.

Choose a linear path  $\theta : [0, 1] \rightarrow \Theta$  between  $A$  and  $B$  such that  $\theta(0) = A$  and  $\theta(1) = B$ . It is component-wise decreasing since  $\beta^j < \alpha^j$ . We have not known that the path  $\theta$  is supported by a continuous family of deformations in the full range. But since there always exists a small deformation by Corollary 2.1.2, we may assume that  $C$  is actually deformable at least in the range  $[0, \omega)$  for some  $0 < \omega \leq 1$ . Then since  $\theta$  is angle decreasing, the family converges strongly to a compact hyperbolic 3-cone-manifold  $C_*$  by Theorem 7.1.1, where the angle set of  $C_*$  is equal to  $\theta(\omega)$ . Then by Corollary 2.1.2, the deformation can extend further. The prolongation of the deformable range can be done up to when  $t$  reaches to 1. Hence we have obtained a continuous family of deformations  $\{C_\theta\}$  for full range of  $\theta$ .  $C_{\theta(1)}$  is homeomorphic to  $C$ . Moreover  $C_{\theta(1)}$  is an orbifold, and hence  $C_{\theta(1)}$  and  $C_{\zeta(1)}$  are isometric by Mostow rigidity. We thus have connected two cone-manifolds  $C$  and  $C_{comp}$  through  $\{C_\theta\}$  and  $\{C_\zeta\}$ .  $\square$

*Proof of Corollary 1.* Suppose we are given two cone-manifolds  $C$  and  $C'$  which are isomorphic. They can be deformed along the same angle decreasing path used in the proof of Theorem to the complete manifold  $C_{comp}$  and  $C'_{comp}$ . The destinations are isometric by Mostow-Prasad rigidity. Then the returning path to  $C$  and  $C'$  must be the same since the cone angle is the only parameter by the local rigidity.  $\square$

*Proof of Corollary 2.* This is now obvious since our family is supported by a path of holonomy representations and one terminal corresponds to the complete structure which is liftable.  $\square$

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